RECONSTRUCTING AGGREGATE DYNAMICS IN HETEROGENEOUS AGENTS MODELS
A MARKOVIAN APPROACH[1]

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The restrictive assumptions imposed by the traditional methods of aggregation prevented so far a sound analysis of complex system of feedback between microeconomic variables and macroeconomic outcomes. This issue seems to be crucial in macroeconomic modelling, in particular for the analysis of financial fragility, as conceived in the Keynesian and New Keynesian literature. In the present paper a statistical mechanics aggregation method is applied to a financial fragility model. The result is a consistent representation of the economic system that considers the heterogeneity of firms, their interactive behaviour and the feedback effects between micro, meso and macro level. In this approach, the impact of micro financial variables can be analytically assessed. The whole dynamics is described by a system of dynamic equations that well mimics the evolution of a numerically solved agent based model with the same features.

Keywords: Financial Fragility, Markov Dynamics, Heterogeneity, Mean-Field Interaction, Master Equation

The Representative Agent (RA) assumption is a methodological shortcut to bypass the problem of dimensionality which arises in heterogeneous agents model. The reasons for dissatisfaction with the RA assumption are well known and have been forcefully

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discussed in Kirman (1992) and Keen (2011). The efforts to overcome the limits of the exact aggregation (Gorman, 1953) led to methods, such as Lewbel (1992), that are still too restrictive in their basic assumptions to realistically depict an economic system.[2]

As a consequence of the dissatisfaction with the RA approach, a few analytical frameworks have been developed to cope with the dimensionality problem mentioned above. One of the most promising methods has been introduced by Duncan Foley and Masanao Aoki who borrowed from statistical mechanics the concept of mean-field interaction and imported it into economics.[3]

In the mean-field interaction approach, agents are classified into clusters or sub-systems according to their state with respect to one particular feature (the so-called micro-state, e.g. the level of production for a firm on a scale of production levels). This clustering determines the characteristics and the evolution of the aggregate (the macro-state, e.g. the total level of output).[4] The focus is not on the single agent, but on the number or fraction of agents occupying a certain state of a state-space at a certain time. These numbers or fractions are governed by a stochastic law, that also defines the functional of the probability distributions of aggregate variables and, if they exist, their equilibrium distributions. The stochastic aggregation is then implemented through master equation techniques, that allow for a description of the dynamics of probability flows among states on a space. These probability flows are originated by the changes in the conditions of agents and determine the aggregate outcomes.[5]

This paper presents an application of mean-field interaction and master equation on a model in which firms are heterogeneous in terms of financial fragility, along the lines of Di Guilmi *et al.* (2010). The degree of financial fragility, modelled *à la* Greenwald and Stiglitz (1993) (GS henceforth), is the clustering device to classify firms and to develop the analytical solution of the model. The

2. For a review on aggregation methods see Gallegati *et al.* (2006) and Di Guilmi (2008).
3. See Foley (1994); Aoki (1996, 2002); Aoki and Yoshikawa (2006). Further developments of these contributions are: Landini and Uberti (2008), Di Guilmi (2008) and Di Guilmi *et al.* (2011).
4. An early economic application of mean-field theory is Brock and Durlauf (2001).
5. Other applications of master equation in economics, besides the works cited above, can be found in Weidlich and Braun (1992) and Garibaldi and Scalas (2010) among others. Alfarano *et al.* (2008) and Alfarano and Milakovic (2009) offers a further contribution, in particular with reference to agent based pricing models.

analytical approximation mimics well the dynamics of a system with a higher order of heterogeneity and provides insights on the interactions among the micro-units in the system. The analytical solution to agent based models is the result of a functional-inferential method which identifies the most probable path of the system dynamics. The method considers the heterogeneity, representing a large number of agents, and the interaction among them, which originates fluctuations of the macroeconomic variables about a deterministic trend. Individual direct interaction is replaced by indirect mean-field interaction between sub-systems, expressed in terms of the transition rates of the master equations. In particular, according to the local approximation method detailed below, an explicit solution for the master equation is obtained. It yields the analytical identification of an ordinary differential equation, which describes the dynamics of the system trend, and a stochastic differential equation, which quantifies the dynamics of the probability distribution of fluctuations.

The successful application of the aggregation method can be a contribution toward the adoption of a realistic new economic paradigm in the direction suggested by Aoki. As shown in the last section, in fact, the numerical simulation of a similar agent based structure is well reproduced by the stochastic dynamics generated by the master equation.[6]

The structure of the paper is the following: first, we specify the hypotheses for the stochastic structure of the system (section 1) and for the firms that compose it (section 2). In section 3, we develop the framework, setting the dynamical instruments needed for aggregation, and solve the model, determining the two equations that drive production trend and business fluctuations. Section 4 presents a further result coming from the solution of the master equation, stressing the relevance of indirect interaction among agents in shaping macroeconomic outcomes. In section 5, some results of computer simulations are presented. Section 6 concludes.

6. On this point see also Chiarella and Di Guilmi (2011).

1. Stochastic structure

The economy is populated by a fixed number N of firms, each indexed by the subscript i. Agents clusters into micro-states according to a quantifiable individual variable. Two micro-states are defined. State 0 denotes agents characterised by a level of a chosen feature above (or equal to) a certain threshold and 1 labels the state of the rest of the population. In each cluster, therefore, there will be a certain number (the so-called occupation number) of agents. The occupation number of cluster j is N^j, $j = 0,1$. The occupation numbers $(N^0(t), N^1(t))$ define the macro-state of the system. The fraction of firms in micro-state j is $n^j = N^j / N$ where $N^0(t) + N^1(t)$. For the sake of tractability, within each cluster individual levels of a certain variable are approximated by their mean-field values, *i.e.* a specific statistic of the distribution of the variable itself.[7] Therefore, within each cluster heterogeneous agents (characterised by different individual levels) are replaced with an homogeneous agent characterised by this statistic (mean field approximation).

The notation adopted uses a continuous time reference because it is more appropriate for complex systems settings, as remarked among others by Hinich *et al.* (2006). Continuous time functionals are appropriate at system level if we assume that the density of discrete points is large enough within a sufficiently small reference interval of time. This is due to the so-called *principle of limiting density of discrete points*, introduced by Jaynes (1957) to match Shannon's entropy with continuous distributions in information and probability theory.[8] For computational necessity, the numerical simulations must refer to discrete time and, accordingly, occupation numbers, as any other observable, become a discrete time stochastic process.

The probability for a firm of being in micro-state 1 is η: $p(1) = \eta$, hence $p(0) = 1 - \eta$. In order to model the probabilistic flow of firms

7. For example, in our simulations, we adopt the median within each group, as specified in section 5.

8. On this topic other interesting references are Smith (1993) and Milakovic (2001). Besides the *principle of limiting density of discrete points*, modelling discrete time observables with continuous time tools is acceptable when the simulation time, say T, is long enough such that the calendar can be partitioned with sufficiently dense adjacent reference intervals of time of order $o(T)$ w.r.t. the calendar. This conjecture is considered as appropriate due to consistency of analytical trajectories from master equations to experimental simulations.

from one microstate to another, transition probabilities and the transition rates must be defined. The functional specification of transition rates, *i.e.* transition probabilities per vanishing reference unit of time, allows the occupation numbers to be modeled with jump Markov processes.

The transition probability is the probability for a firm to switch from one microstate to the other in a given instant. The transition probability of moving from 0 to 1 is ζ while ι indicates the probability of the opposite transition. The transition rates quantify the probability of observing a jump of one agent from one microstate to another, conditional upon the initial microstate through time. A transition rate is then given by the probability of a firm changing state weighted by the probability of being in one particular starting state. With reference to state 1, the transition rate for entry (from state 0 into state 1) is indicated with λ while the one for exit (from state 1 to state 0) is γ, defined as follows:

$$\begin{aligned} \lambda &= \zeta(1-\eta) \\ \gamma &= \iota\eta \end{aligned} \tag{1}$$

This representation is phenomenological. Indeed, it allows either for λ, γ and η to be constants or functionals of some state variable.[9] In case of only two micro-states, N being constant through time, the attention is focused on only one occupation number (for instance N^1) to characterise the macro state of the entire economy in a given instant, $1 \leq N_k \leq N$: a realisation of the stochastic process $N^1(t)$ on its support is denoted with $N^1(t) = N_k$.

The transition rates determine the probability of observing a certain occupation number at the aggregate level, *i.e.* a certain macrostate of the system. Being $N^1(t) = N_k$, within the length of a vanishing reference unit of time $\Delta \rightarrow 0^+$, the expected number of transitions into the macrostate N^1 is $\lambda(N - N_k)$ while the expected number of transitions from macrostate N^1 is γN_k; therefore, the transition rates can be written as follows

$$\begin{aligned} b(N_k) &= P\left(N^1(t+\Delta) = N_{k+1}(t') \,\middle|\, N^1(t) = N_k(t)\right) = \lambda(N - N_k) \\ d(N_k) &= P\left(N^1(t+\Delta) = N_{k-1}(t') \,\middle|\, N^1(t) = N_k(t)\right) = \gamma N_k \end{aligned} \tag{2}$$

9. In Appendix A the stochastic model results are discussed for both cases.

where b and d indicate, respectively, "births" ($N_k \rightarrow N_{k+1}$) and "deaths" ($N_{k-1} \leftarrow N_k$) rate functions of the stochastic process and $t' - t = \Delta$.

2. Firms

This section presents the assumptions for the microeconomic units of the system. The approach is the one pioneered by GS, and implemented in a heterogeneous agents framework by Delli Gatti *et al.* (2005). If not otherwise specified, variables indicated by small letters refer to single firms while symbols in capital letters stand for aggregate quantities, within the state if followed by the superscripts 0 or 1 and economywide otherwise.

2.1. Financial fragility as a clustering device

We assume that financially constrained firms are subject to iid shocks to revenue and, therefore, they run the risk of bankruptcy if revenue fall short of pre-incurred costs. In this setting the optimal scale of activity for the firm is constrained by its net worth due to bankruptcy risk. The firm's probability of bankruptcy depends upon its equity ratio, *i.e.* the ratio of net worth to assets.

In the present paper this approach has been followed in a somewhat stylised way. The economy is populated by a fixed number N of firms which agglomerate into clusters depending on the level of individual equity ratio $\alpha_i = a_i / k_i$, $i = 1, 2, \ldots N$, where a_i is net worth[10] and k_i total assets (physical capital). The threshold $\overline{\alpha}$ divides the populations of firms in two clusters: firms in state 0 (whose occupation number is N^0), characterised by $\alpha_i \geq \overline{\alpha}$, are financially robust while firms in state (whose occupation number is N^1). characterised by $\alpha_i < \overline{\alpha}$, are financially fragile and exposed to the risk of bankruptcy. Within each cluster, individual levels of the equity ratio are approximated by their mean-field values α^0 and α^1 respectively.

In order to keep the number of firms N constant, each bankrupted firm is replaced by a new one which, by assumption, enters the system in state 1. The probability of being fragile is η

10. Equity or own capital are assumed synonyms of net worth.

while μ denotes the probability of bankruptcy—*i.e.* of exiting from the economy. Hence the rate of exit from the system is $\mu\eta$. Of course, due to the one-to-one replacement assumption, $\mu\eta$ represents also the rate of entry into the system.

2.2. Technology, costs and prices

Each firm employs physical capital as the only input in production. Therefore, the production function of the i-th firm is:

$$q_i (t) = (2k_i (t))^{1/2} \tag{3}$$

and the capital requirement function is:

$$k_i (t) = 1/2\ (q_i (t)^{2} \tag{4}$$

Firms can finance capital with previously retained profits (net worth). When internal funds are not sufficient, firms resort to loans: $b_i (t) = k_i (t) - a_i (t)$. Debt commitments in real terms are $rb_i (t)$, where r is the real interest rate.[11] For the sake of simplicity the interest rate is constant and uniform across firms.

The firm has no market power (it is a price taker) but is operating in an uncertain environment. The price $P_i (t')$ of the t'-th firm at time t'—*i.e.* when the output is actually sold—is equal to the average or market price $P(t)$ at time t—i.e. when the output is produced and ready for sale—subject to an idiosyncratic multiplicative shock $\tilde{u}_i (t')$:

$$P_i (t') = \tilde{u}_i (t') P(t) \tag{6}$$

The random variable $\tilde{u}_i (t') \to U(u_0, u_1)$ s.t. $u_1 > u_0 > 0$ and $\mathbb{E}(\tilde{u}_i) = 1$. Its support can be any positive neighbourhood of 1: in this paper it has been chosen to set $\tilde{u}_i (t')$ within $[u_0 = 0.75;\ u_1 = 1.25]$.[12]

2.3. Profit, net worth and bankruptcy

The law of motion of net worth (in real terms) is:

$$a_i (t') = a_i (t) + \pi_i (t') \tag{7}$$

11. By hypothesis, the return on own capital is equal to the interest rate r, so that the firm's financing costs are:

$$r(b_i (t) + a_i (t)) = rk_i (t) \tag{5}$$

12. Due to the normalisation procedure detailed below, the choice of the support for $\tilde{u}_i$ does not affect probabilities.

where $\pi_i\ (t')$ is profit (in real terms):

$$\pi_i(t') = \tilde{u}_i(t')q_i(t) - rk_i(t) \tag{8}$$

A firm goes bankrupt when $a_i\ (t')$ reaches the zero threshold, *i.e.* when

$$\pi_i\ (t') = -a_i\ (t) \tag{9}$$

Substituting (8) into (9), and solving for $\tilde{u}_i(t')$, the bankruptcy threshold level of the shock is

$$\bar{u}_i(t') \equiv \frac{rk_i(t) - a_i(t)}{q_i(t)} \tag{10}$$

Notice that, by construction, the threshold level of the shock occurring at t' is a function of variables defined at time t'. If the shock $\tilde{u}_i(t') \leq \bar{u}_i(t')$, then equity becomes negative (or zero) and the firm goes bankrupt.

Since $\frac{a_i(t)}{q_i(t)} = \alpha_i(t)\frac{k_i(t)}{q_i(t)}$, and recalling (4), Equation (10) reads as:

$$\bar{u}_i(t') = \frac{q_i(t)}{2}\left[r - \alpha_i(t)\right] \tag{11}$$

The random variable $\tilde{u}_i$ has support [0.75;1.25], therefore, denoting with F the cdf of $\tilde{u}_i(t')$, the probability of bankruptcy μ_i for firm i is

$$\mu_i(t) = F(\bar{u}_i(t)) = \frac{\bar{u}_i(t) - 0.75}{0.5} = 2\bar{u}_i - 1.5 \tag{12}$$

Every firm which goes bankrupt has to bear bankruptcy costs $C_i\ (t)$, non-linearly increasing with firm size,

$$C_i(t) = c(q_i(t))^2 \quad 0 < c < 1 \tag{13}$$

As discussed by Greenwald and Stiglitz (1990), bankruptcy costs hold to the borrower. They are due to legal and administrative costs incurred during the bankruptcy procedure and to the reputational costs for the managers of a firm which goes bankrupt. These reputational costs are assumed to be increasing with the scale of production.

2.4. Output

Following GS, we assume that at time t the firm (optimally) decides the quantity to produce which will be sold at t' in conditions of uncertainty concerning the sale price. Therefore, the problem of the firm at t consists in maximising an objective function V which is equal to expected profits at t', net of bankruptcy costs, subject to the production function (3):

$$\max V = \mathbb{E}\{\tilde{u}_i(t')q_i(t) - rk_i(t) - C_i(t)\mu_i\} \tag{14}$$

$$s.t. \quad k_i(t) = \frac{1}{2}(q_i(t))^2 \tag{15}$$

Since $\mathbb{E}(\tilde{u}_i(t')) = 1$, assuming that agents consider the expected probability of bankruptcy at time t' equal to the one at time t, the problem above boils down to the following:

$$\max V = q_i - rk_i - C_i\mathbb{E}(\mu_i) \tag{16}$$

$$s.t. \quad k_i = \frac{1}{2}(q_i)^2 \tag{17}$$

where the time index has been removed to simplify notation.

Firms in state 0 know that their probability of bankruptcy is $\mathbb{E}(\mu_0) = 0$. Hence, for financially robust firms, the problem is:

$$\max V_0 = q_0 - rk_0 \tag{18}$$

$$s.t. \quad k_0 = \frac{1}{2}(q_0)^2 \tag{19}$$

which solves with $q^0 = r^{-1}$ being r given and q^0 constant through time. Financially fragile firms know that they run the risk of bankruptcy. Due to the mean-field approximation, the probability of bankruptcy for firms in state 1 is constant across agents. Hence the optimisation problem becomes:

$$\max V_1 = q_1 - rk_1 - c(q_1)^2\mu \tag{20}$$

$$s.t. \quad k_1 = \frac{1}{2}(q_1)^2 \tag{21}$$

and the solution is

$$q^1 = (r + 2c\mu)^{-1} \tag{22}$$

Note that μ is indeed defined at time t and time dependent so that also q^1 is time dependent. Aggregate production is:

$$Y = Y^1 q^1 + N^0 q^0 \tag{23}$$

where N^1 and N^0 are the previously introduced occupation numbers. Plugging the above obtained results into this definition, aggregate output can be expressed as

$$Y = \frac{N^1}{r + 2c\mu} + \frac{N^0}{r} \tag{24}$$

From Equation (24), it is clear that business fluctuations are driven by (i) the probability of bankruptcy μ and (ii) the dynamics of the occupation numbers. The impact of financial fragility on the aggregate may be better appreciated by reformulating Equation (24) as follows

$$Y = \frac{N}{r} - \xi N^1 \tag{25}$$

where $\xi = \frac{1}{r}\left(1 + \frac{r}{2c\mu}\right)^{-1}$.

While N and r are given, each factor in the product ξV^1 is defined at time t and time dependent. N^1 can be considered a macroeconomic indicator of the financial fragility of the system; in (25) it is weighted by ξ, which is a function of the probability μ. Therefore, the dynamics of aggregate production appears to be determined by the micro and macro level of financial distress of the economy.

2.5. Transition probabilities

The probability of bankruptcy μ_i can be expressed as

$$\mu_i = F(\bar{u}_i) = \frac{\bar{u}_i - 0.75}{0.5} = 2\bar{u}_i - 1.5 \tag{26}$$

By assumption, only firms in state 1 are exposed to the risk of bankruptcy. It is expected firms lumped in cluster 1, the group of financially fragile firms, have the following bankruptcy threshold:

$$\bar{u}_1 \equiv \frac{q^1}{2}(r - \alpha^1) \tag{27}$$

Hence $\quad \mu = F(\bar{u}_1) = 2\bar{u}_1 - 1.5 = q^1(r - \alpha^1) - 1.5$

According to equation (26), it is possible to quantify the equity ratio threshold $\bar{\alpha}$, which is the minimum level of the equity ratio that ensures the firm's survival (*i.e.* $\mu = 0$),[13] and can be expressed as

$$\bar{\alpha} = r - \frac{1.5}{q^1} \tag{28}$$

Since q^1 is time dependent, the threshold $\bar{\alpha}$ also evolves over time.

The transition probabilities ζ (*i.e.* the probability of moving from 0 to 1) and ι (*i.e.* the probability of moving from 1to 0) can be expressed as variables depending on the price shock $\tilde{u}_i(t)$, with the appropriate critical values $\bar{u}_\zeta(t)$ and $\bar{u}_\iota(t)$

$$\begin{aligned} \tilde{u}_i(t) &\leq \frac{q^0(t)}{2}\left[r + \bar{\alpha}(t) - \alpha^0(t)\right] \equiv \bar{u}_\zeta(t) \\ \tilde{u}_i(t) &> \frac{q^1(t)}{2}\left[r + \bar{\alpha}(t) - \alpha^1(t)\right] \equiv \bar{u}_\iota(t) \end{aligned} \tag{29}$$

The explicit formulation for transition probabilities is therefore

$$\zeta(t) = p(\tilde{u}(t) \leq \bar{u}_\zeta(t)) = 2\bar{u}_\zeta(t) - 1.5 \tag{30}$$

$$\iota(t) = 1 - p(\tilde{u}(t) \leq \bar{u}_\iota(t)) = -2\bar{u}_\iota(t) + 2.5 \tag{31}$$

3. Dynamic analysis and solution

This section introduces the master equation, which is the fundamental tool in the analytical solution process, and the main result of its asymptotic solution.

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13. It is now straightforward identifying an upper bound for the total credit demand $B = B^0 + B^1$, where B^0 and B^1 are the total demands for each group of firms. Given the optimal levels of capital for each cluster of firms, namely k^1 and k^0, the quantity of credit demanded reaches its maximum when α^1 and α^0 reach their minimum. Note that α^1 cannot go below $r - 2.5/q^1$, at which value μ becomes equal 1. By definition, the minimum level for α^0 is $\bar{\alpha} = r - 1.5/q^1$. For these values it follows that:

$$\lim_{\alpha^0 \to \bar{\alpha}} B^0 = \frac{(q^0)^2}{2}\left(1 - r + 1.5/q^1\right)$$
$$\lim_{\alpha^1 \to \min(\alpha^1)} B^1 = \frac{(q^1)^2}{2}\left(1 - r + 2.5/q^1\right)$$

Consequently, the demand of credit must be smaller than or equal to:

$$\max(B) = N^0\left[\frac{(q^0)^2}{2}\left(1 - r + \frac{1.5}{q^1}\right)\right] + N^1\left[\frac{(q^1)^2}{2}\left(1 - r + \frac{2.5}{q^1}\right)\right]$$

that cannot grow indefinitely since $q^0 = 1/r$ and $q^1 < q^0$ as shown below.

3.1. Aggregate dynamics

The solution of the model requires the specification of aggregate output dynamics. As shown by equations (24) and (25), aggregate output depends on a stochastic process, whose outcome is given by the occupation numbers N^0 and N^1. It is assumed that the stochastic process is a jump Markov process and its macro dynamics is analytically explored by means of the master equation, *i.e.* a differential equation that describes the dynamics of the probability distribution of a system of agents over its state space through time. The master equation can be primarily specified as a balance flow equation between probability inflows and outflows in and from a generic macro-state.

The state variable $N^1(t) = N_k$ is the number of fragile firms, those in state 1. The variation of probability in a vanishing reference unit of time is

$$\frac{dP(N_k,t)}{dt} = b(N_{k-1},t)P(N_{k-1},t) + d(N_{k+1},t)P(N_{k+1},t) + \\ -\{[(b(N_k,t)+d(N_k,t))P(N_k,t)]\} \tag{32}$$

with boundary conditions:

$$\begin{cases} P(N,t) = b(N^1)P(N^1-1,t) + d(N)P(N,t) \\ P(0,t) = b(1)P(1,t) + d(0)P(0,t) \end{cases} \tag{33}$$

The variation of probability defined in the equation above is defined as the sum of inflow-births from N_{k-1} and inflow-deaths from N_{k+1} less outflows from N_k due to births and deaths. Finally, the boundary conditions ensure a consistent value for the probability $P(N_k)$. Therefore, in order to identify the dynamics of firms and production, Equation (32) must be solved.

3.2. Master equation's solution: stochastic dynamics of trend and fluctuations

As shown by (Gardiner, 1985; Risken, 1989), a direct solution of the master equation is possible only under restrictive assumptions. Inspired by van Kampen (2007), Aoki (1996, 2002) and Aoki and Yoshikawa (2006) suggest a method to overcome this problem which consists in splitting the control variable into the drift and diffusion components of the underlying process.[14] The application of this method appears of particular interest in this context as it allows to analytically identify both the trend and the fluctuations distribution of aggregate production. More precisely, the fraction of firms in state 1 in a given instant is assumed to be determined by its expected value (m), the drift, and an additive fluctuations component of order $N^{1/2}$ around this value, that is the spread:

$$N_k = Nm + \sqrt{N}s \tag{35}$$

Once the master equation has been modified accordingly, that is in terms of s rather than of N_k, it can be solved using approximation methods. As shown in Appendix A, the asymptotically approximated solution of the master equation is given by the following system of coupled equations:

$$\frac{dm}{dt} = \zeta m - (\zeta + \iota)m^2 \tag{36}$$

$$\frac{\partial Q(s)}{\partial t} = \left[2(\zeta + \iota)m - \zeta\right]\frac{\partial}{\partial s}(sQ(s)) + \frac{\left[\zeta m(1-m) + \iota m^2\right]}{2}\left(\frac{\partial}{\partial s}\right)^2 Q(s) \tag{37}$$

where $Q(s(t),t) = P(N^1(t),t)$ is substituted into (32) to reformulate the master equation as a function of the spread s. Equation (35) is an ordinary differential equation which displays a logistic dyna-

14. The authors are aware that this method presents some drawbacks. First of all, van Kampen's method develops a local approximation suitable to be applied only when the underlying observable has a unimodal distribution, as the case under study in this paper. Secondly, by allowing for a second order approximation it ends up with a Fokker-Planck equation which solves into a Gaussian distribution for fluctuations: in the present paper it is shown that this is not the distribution for the state variable but only for its spreading fluctuations about the drift. Thirdly, when fluctuations are not of the order of the square root of N, higher order moments might not vanish asymptotically and thus leading to non-Gaussian distributions. This last aspect can be found when dealing with microscopic models grounded on global interactions, which is not the case under study, or when mean-field approached are not very suitable, see Castello *et al.* (2006) and Stauffer *et al.* (2006) on this issue. Despite these limits, this method is adopted for different reasons. First of all it is relatively easy to handle, as shown in the present paper. In particular, if one is interested in macroscopic dynamics of a given quantity or an aggregation procedure, it does not require to provide a solution to the master equation in terms of the probability distribution of the state variable. The aim is to find the equations for the drift and spread only. Secondly, it allows for a complete description of the stochastic aggregate dynamics in terms of transition rates and related parameters, such as transition probabilities at micro-level, which can be analytically obtained from the underlying agent based model. Third: if one is allowed to assume the van Kampen's ansatz (34) and mean-field interaction, rather than global, is considered, then the needed condition for a suitable second order approximation are met. The method is here developed following Landini and Uberti (2008) and Di Guilmi *et al.* (2011).

mics for the drifting component. Equation (36) is a second order partial differential equation, known as the Fokker-Planck equation, that drives the density of the spreading component s. The dynamics converge to the steady state values: setting the l.h.s. of (35) to 0, the stable steady-state value for m is

$$m^* = \frac{\zeta}{\zeta + \iota} \tag{37}$$

Then, by integration of (35) with an initial condition $m(0) = m_0$ we get:

$$m(t) = \frac{m_0 \zeta e^{\zeta t}}{\zeta + m_0 (\zeta + \iota)\left(e^{\zeta t} - 1\right)} \tag{38}$$

This equation describes the dynamics of the fraction m of firms occupying state 1 at each point in time. It is fully dependent on transition rates. The stationary solution of the equation for the spread component (see Appendix B) yields the distribution function $\overline{Q}$ for the spread s, thus determining the probability distribution of fluctuations:

$$\overline{Q}(s) = C \exp\left(-\frac{s^2}{2\sigma^2} \right) : \sigma^2 = \frac{\zeta \iota}{(\zeta + \iota)^2} \tag{39}$$

which looks like a Gaussian density, dependent only on transition probabilities. Given the relationship among m and total production, the dynamics of our economy is now fully described by having at hands a differential equation for output dynamics, its equilibrium value, and a probability function for business fluctuations around the trend.

4. Interaction and output dynamics: the stochastic financial contagion

This section shows how the transition rates provide a functional representation of the interaction of firms within each cluster and of the feedback effects between the macro and the micro-level of our stylised economy. The first subsection proposes and endogenous formulation for the probability η, which makes possible a reinterpretation of the formula for aggregate output and the transition rates, as illustrated in the second subsection.

4.1. Stationary points and equilibrium probability

An important result derived by the asymptotic solution of the master equation concerns the theoretical probability η of being in state 1. In particular it is possible to identify a functional form that quantifies the impact of indirect interaction among firms. By definition, the steady-state condition implies that the probability of in-flows is equal to the probability of out-flows for all possible states. Analytically it means a null value for the r.h.s. of the master equation. This condition is defined as detailed balance.[15] Provided that detailed balance holds for each pair of macro-states, Appendix C shows that the stationary probability for a given macro state $P^e(N_k)$ is

$$P^e(N_k) = P^e(N(0))\left(\frac{\iota}{\zeta}\right)^{N_k} \binom{N}{N_k} \prod_{h=1}^{H} \frac{\eta(N-N_h)}{(1-\eta(N_h))} \tag{40}$$

The probability $P^e(N_k)$ can be also expressed in Gibbs form,[16] and a Gibbs functional form for the probability η is

$$\eta(N^1) = N^{-1} e^{\beta g(N^1)} \tag{41}$$

where:

$$\beta(t) = \ln\left(-\frac{y^1(t)-\bar{y}(t)}{y^0(t)-\bar{y}(t)}\right)\left(y^1(t)-y^0(t)\right)^{-1}$$
$$g(N^1) = -\frac{1}{2\beta}\ln\left(\frac{N^1}{N-N^1}\right) = \frac{y^0-y^1}{2} \tag{41'}$$

The symbol $\bar{y}(t)$ stands for the average production $Y(t)/N$. The probability of being in state 1 in a given instant depends on three factors: the number of firms already occupying the state, N^1; the parameter β, which measures the impact on total output of the relative financial distress of firms; the function $g(N^1)$, which quantifies the average difference in the optimal levels of production. The circular feedback effects are displayed by Equation (41): the macro-to-micro effect captures the link of the behaviour of a firm

15. It is worth stressing that the detailed balance does not imply that agents do not switch between the micro-states, but that inflows and outflows for each micro-state balance out.
16. The equivalence is demonstrated by the Hammersley and Clifford theorem which states that for each Markov random field there exists one and one only Gibbs random field (Clifford, 1990).

to the state of the economy; the bottom-up or micro-to-macro effect, on the other hand, determines the aggregate performance by the number of firms with lower output and by the relative difference in optimal outputs, captured by $g(N^1)$ and β. Therefore, by means of equation (41), the whole dynamics of the system can be interpreted as the result of indirect interaction among firms and of the feedback effects between macro, meso and micro level.

4.2. Output dynamics

Making use of equations (25) and (37), the steady-state value of aggregate production, Y^e, can be expressed as:

$$Y^e = N\left[\frac{1}{r} - \frac{\zeta}{\zeta+\iota}\frac{2c\mu}{r(r+2c\mu)}\right] = N\left[\frac{1}{r} - \frac{\zeta}{\zeta+\iota}\left(y^0 - y^1\right)\right] \tag{42}$$

Equation (42) highlights two factors that influence production dynamics: the difference between firms' optimal production levels and the transition rates. The first component is determined by the exogenous parameter c, that reflects institutional conditions, and by the probability of bankruptcy μ, that is the result of the relative financial condition of financially distressed firms, being a function of the difference between their equity ratio α^1 and the "safety" level $\overline{\alpha}$.

The transition rates component is the result of a micro factor (the relative financial conditions of the two types of firms) and of a macro factor (the general financial situation of the system, revealed by the number of firms in each state), as shown by equations (1) and (30). The formulation of λ and γ under detailed balance condition helps in clarifying further the point. Substituting equations (30) and (41) in equations (1), we obtain:

$$\lambda = \left(N^{-1}e^{\beta g(N^1)}\right)\left(2\overline{u}_\zeta - 1.5\right) \tag{43}$$

$$\gamma = \left[1 - \left(N^{-1}e^{\beta g(N^1)}\right)\right]\left(-2.5\overline{u}_\iota + 1.5\right) \tag{44}$$

The micro factor is quantified by $\overline{u}_\zeta$ and $\overline{u}_\iota$, that, as shown in equations (29), reflect the difference between $\overline{\alpha}$ and the mean-field variables α^1 and α^0. This effect is amplified by the macro factor that, in turn, is dependent on the occupation numbers and on the relative difference in optimal levels of production. The joint effect of these variables gives rise to a mechanism that can be defined as stochastic financial contagion, given that a worsening in micro financial conditions raises both the probability of bankruptcy and the probability of entering state 1.

The solution reveals that the dynamics of the economy is dependent on the distribution of agents and on its evolution. Given the inherent uncertainty of these dynamics, all the functional relationship are expressed as probability functions. Therefore, the dynamics of the system appears to be fully stochastic, and the steady-state level of production cannot be considered as a natural equilibrium.

5. Simulations

In order to visualize the actual dynamics of the system and check the reliability of the stochastic approximation, Monte Carlo simulations have been performed. The agent based model have been simulated with fully heterogeneous firms according to the hypothesis detailed in section 2. Then mean-field variables α^1 and α^0 as the medians of the equity ratios within each cluster have been calculated. These values are the input of the stochastic dynamics procedure, performed according to the structure of section 1. The simulation has been repeated 1000 times, drawing a new set of random numbers for each replication. The number of firms is $N = 300$ and the parameter c is set equal to 1.

To appreciate the volatility endogenously generated by the system, figure 1 displays the symmetric dynamics of low-equity firms and aggregate production for a single replication. The convergent evolution of n^1 is driven by equation (38) with fluctuations around the trend distributed according to (39). Its dynamics fully explains the growth of aggregate production and the business fluctuations. The higher volatility in the series of output is due to the shocks in price.

Figure 2 compares the agent based results with its stochastic approximation in the initial stages of the adjustment process. Agent based trend dynamics are well mimicked by the stochastic approximation. The fluctuations generated by the two procedures cannot match, as in the latter they are the outcome of a random variable. Nevertheless, the amplitudes of volatility are comparable.

The result is satisfactory as the average variance of the time series is .0068 for agent based and .0063 for the stochastic approximation.

Figure 1. Trends and fluctuations for value of aggregate production (left scale) and n^1 (right scale). Single replication

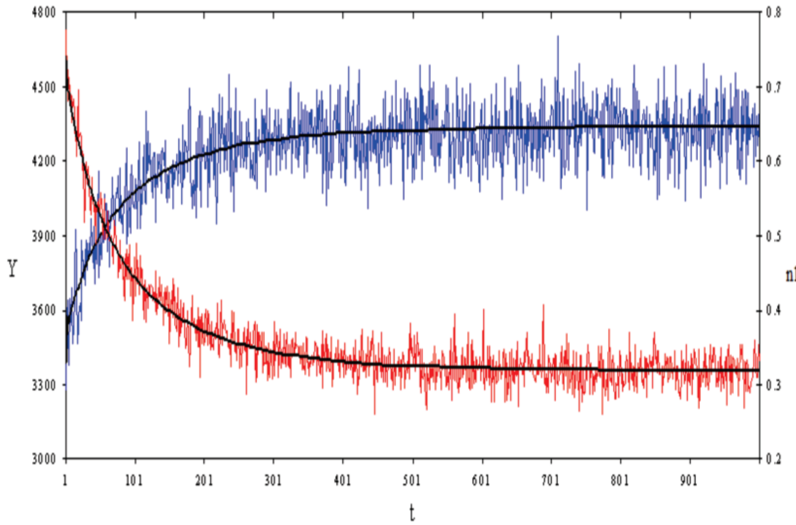
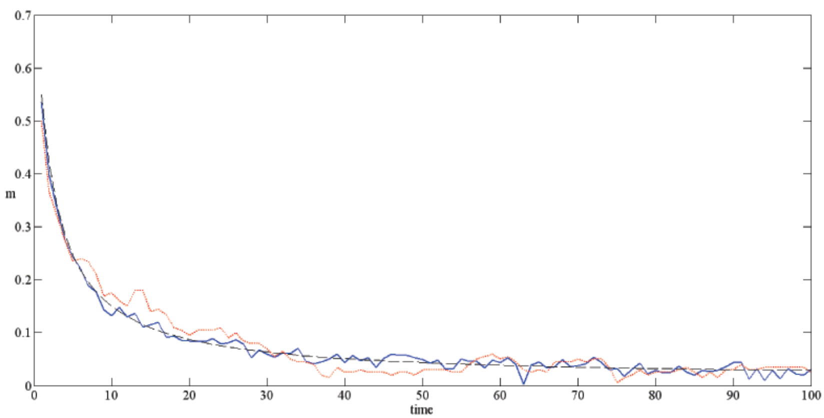


Figure 2. Dynamics for n^1 for agent based simulation (red dotted line) and stochastic approximation (black dashed line for trend and continuous blue line for fluctuations). Single replication



The dynamics of the two series obtained by the Monte Carlo simulation over 1000 replications are displayed in figure 3. They overlap for almost all the periods, although the adjustment process to the steady state is shorter for the stochastic approximation. Their significant correlation is .96. Thus, the stochastic dynamics proves reliable for an analytical representation of more complex and diversified structure. The simplification to the two states approximation does not seem to reduce the accuracy of the solution.

Figure 3. Dynamics for n^1 for agent based simulation (red dotted line) and stochastic approximation (black continuous line). Monte Carlo simulation with 1000 replications

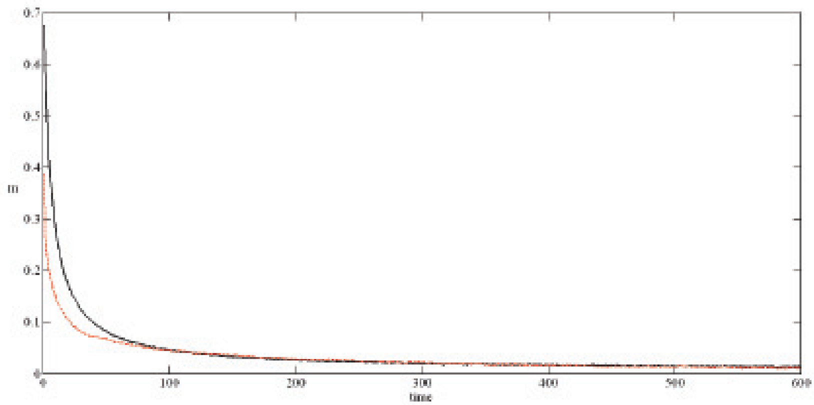
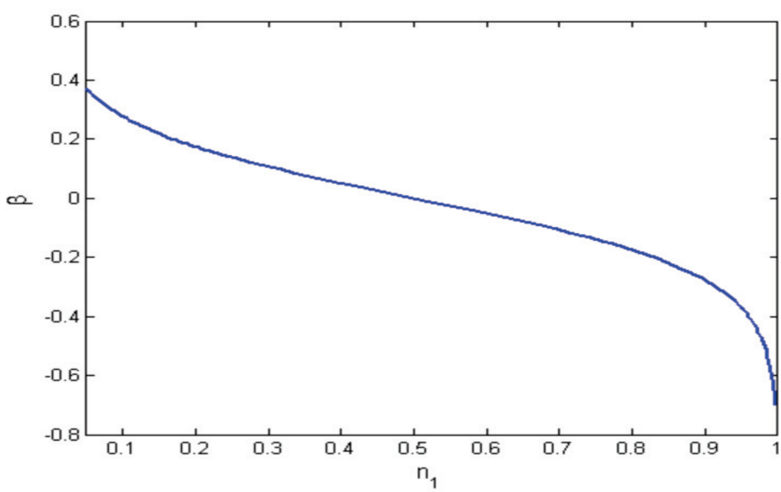
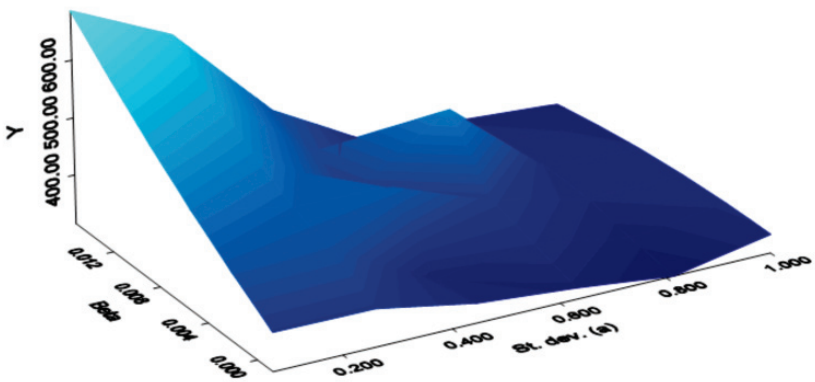


Figure 4. Relationship between β and n^1



The variable β, which enters the definition of transition rates, is inversely related to the number of financially fragile firms, as shown by figure 4. Hence, it represents an inverse index of the systemic financial fragility. According to Equation (42), the aggregate output is expected to be higher for lower level of N^1 and, thus, higher β. This result is confirmed by the simulation and illustrated in figure 5. The same graph reveals that the performance of the economy is also dependent on the shape of the distribution of the net worth, as lower levels of standard deviation for a appear to be associated to a larger aggregate production.

Figure 5. Countour plot of the aggregate output as a function of the standard deviation of net worth distribution and β



6. Concluding remarks

This work proposes a solution to the problem of the aggregation of heterogeneous agents in a dynamical context by applying a method which analytically identifies the components of macroeconomic dynamics, namely, trend and fluctuations. It is worth stressing that the long run steady-state of production cannot properly be defined as natural equilibrium. From the methodological point of view, the main contribution of the present work is the identification of a differential equation for trend and a probability distribution function for the fluctuations of the aggregate production by means of the asymptotic solution of the master equation. All the variables that appear in these two formulations are endogenous and provide an analytical representation of the interaction among agents and the feedback effects that arise among the different levels of aggregation within the system. In particular, both the probability for a firm to reduce its production as a consequence of the risk of failure and the actual probability of bankruptcy are dependent on the financial distress of the other firms in the system, measured by the number of firms with low equity ratio and by the mean-field approximations of the equity ratios. Aggregate production is itself dependent on the ratio among debt and equities of each firm, and this gives rise to feedback effects between micro and macro levels of the system. The overall effect can be defined as stochastic financial contagion.

This methodology appears as particularly suitable for models where the micro financial variables have a relevant impact on the macroeconomy. In such a way, the modelling of the links among financial fragility, business cycles and growth dynamics can be consistently microfounded, taking into account the heterogeneity of firms' financial variables and the interaction among agents and between agents and the macro level of the system. However, the actual range of application of this body of tools extends to all the contexts in which the heterogeneity of agents and their interaction cannot be neglected or reduced in order to represent, *e. g.*, the efficacy of an economic policy measure or the transmission mechanism of a shock. All in all, the whole of macroeconomics.

The limitation to the heterogeneity does not seem to impact on the performance of the model that proves capable to replicate the behaviour of an analogous agent based model, with no restrictions on the heterogeneity of firms.

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Appendix A

This appendix develops the method involved to reach a mean-filed system of coupled equations for the drift and the spread of the state variable in the master Equation (32). According to (34) for a fixed it follows that $N^1(t) = N_k \in [1, N]$

$$s(t) = (N_k - Nm(t))N^{-1/2} \quad \text{(A.1)}$$

Accordingly, the master equation (ME) (32) can be rewritten a function of the state variable s. The fact that N_k is fixed does not mean that it is constant but just that we focus out attention on it as a specific realization of $N^1(t)$; Accordingly, from (A.1) it follows that

$$\frac{ds}{dt} = -\sqrt{N}\frac{dm}{dt} \quad \text{(A.2)}$$

hence the l.h.s. of (32) reads as

$$\frac{dP(N_k)}{dt} = \frac{\partial Q(s)}{\partial t} - \sqrt{N}\frac{dm}{dt}\frac{\partial Q(s)}{\partial s} = \frac{dQ(s)}{dt} \quad \text{(A.3)}$$

being $P(N^1(t) = N_k) = Q(s(t))$. In order to find a suitable expression for the r.h.s. of (32), transition rates are written as follows

$$\begin{aligned} b(N_{k-\theta}) &= \lambda[N-(N_k-\theta)] = \lambda N\left\{1-\left[m+\frac{1}{\sqrt{N}}\left(s-\frac{\theta}{N}\right)\right]\right\} \\ &= \lambda N\left[1-\left(m+\frac{1}{\sqrt{N}}s_{(-\theta)}\right)\right] = b(s_{(-\theta)}) \end{aligned} \quad \text{(A.4)}$$

$$\begin{aligned} d(N_{k+\theta}) &= \gamma[N_k+\theta] = \gamma N\left[m+\frac{1}{\sqrt{N}}\left(s+\frac{\theta}{\sqrt{n}}\right)\right] \\ &= \gamma N\left(m+\frac{1}{\sqrt{N}}s_{(+\theta)}\right) = d(s_{(+\theta)}) \end{aligned} \quad \text{(A.5)}$$

where $\theta = 0$ means outflow and $\theta = 1$ inflow, consistently with the phenomenological ME (32).

The lead (+) and lag (–) operators are defined as

$$f_{(\pm)}(N_{k\pm\theta})) = L^{\pm}[f_{(\pm)}(N_{k\pm\theta})]_{\theta=0} = \sum_{z=1}^{\infty}\frac{(\pm N^{-1/2}\partial_s)^z}{z!}[g_{(\pm)}(s_{(\pm\theta)})]_{\theta=0} \quad \text{(A.6)}$$

where

$$f_+(N_{k+\theta}) = d(N_{k+\theta})P(N_{k+\theta}) = d(s_{(+\theta)})Q(s_{(+\theta)}) = g_+(s_{(+\theta)}) \quad \text{(A.7)}$$

$$f_{-}(N_{k-\theta}) = b(N_{k-\theta})P(N_{k-\theta}) = b(s_{(-\theta)})Q(s_{(-\theta)}) = g_{-}(s_{(+\theta)}) \quad \text{(A.8)}$$

Hence it follows that

$$d(N_{k+1})P(N_{k+1}) = L^{+}[d(N_k)P(N_k)] = d(s_{(+1)})Q(s_{(+1)}) = L^{+}[d(s)Q(s)] \quad \text{(A.9)}$$

$$b(N_{k-1})P(N_{k-1}) = L^{-}[b(N_k)P(N_k)] = b(s_{(-1)})Q(s_{(-1)}) = L^{-}[b(s)Q(s)] \quad \text{(A.10)}$$

Therefore, by using (A.9) and (A.10) into the r.h.s. of (32) it follows that

$$\frac{dQ}{dt} = [L^{+} - 1](d(s)Q(s)) + [L^{-} - 1](b(s)Q(s)) \quad \text{(A.11)}$$

which is the ME to be solved. The solution is approximated as (A.6) involves Taylor's polynomials to approximate probability flows about N_k.

Rescaling time as $t = N\tau : [t] \neq [\tau]$, with a second order approximation it follows

$$\partial_t Q - \sqrt{N}\dot{m}\partial_s Q = N^{-1/2}\partial_s[\rho_{-}(s)Q(s)] + N^{-1}\frac{1}{2}\partial_s^2[\rho_{+}(s)Q(s)] \quad \text{(A.12)}$$

where, for notation convenience, t stands for r. Expression (A.12) is a Fokker-Planck equation, equivalent to the approximation one gets with the Kramers-Moyal expansion if Pawulas' theorem does not allow for a closed form solution (see Risken, 1989; Gardiner, 1985; Di Guilmi *et al.*, 2011), and coefficients are given by

$$\rho_{\pm}(s) = d(s) \pm b(s) \quad \text{(A.13)}$$

Case 1. If transition rates in (2) have birth (λ) and death (γ) rates constant through time, by substituting (A.13) into (A.12) according to (A.4) and (A.5) with $\theta = 0$, after having computed derivatives and collected terms with powers of N, it happens that as $N^{-p/2} \to 0$ as $N \to \infty$ $\forall p \geq 2$ hence, by using the polynomial identity principle, it can be found that

$$\begin{cases} (i) \quad \dot{m} = \lambda - (\lambda + \gamma)m \\ (ii) \quad \partial_t Q(s) = [\lambda + \gamma]\partial_s(sQ(s)) + [\lambda - (\lambda + \gamma)m]\frac{1}{2}\partial_s^2 Q(s) \end{cases} \quad \text{(A.14)}$$

The (A.14–*i*) in the mean-field system gives the so called macroscopic equation: its solution provides the most probable drifting path trajectory for $\langle N^1(t)/N\rangle$. The (A.14–*ii*) is the Fokker-Planck equation for the probability distribution of spreading fluctuations about the drift. Both admit a closed form solution allowing for a solution of the ME (A.11) equivalent to (32).

Case 2. In the case of this model, the transition rates have birth and death rates which change over time. Therefore, the following externality functions are introduced in order to model their evolution

$$\lambda(t) = \Lambda(N^1(t)) = \zeta(t)\left[\frac{N^1(t)}{N}\right] \tag{A.15}$$

$$\gamma(t) = \Gamma(N^1(t)) = \iota(t)\left[\frac{N^1(t)-1}{N}\right] \tag{A.16}$$

Therefore, since $N^1(t) = N_k$, (A.15) and (A.16) can be substituted into (A.4) and (A.5) with $\theta = 0$ to get an expression for (A.13) with the modification of transition rates just highlighted. Subsequently, after the derivatives have been computed, the terms with the same order of powers for N are collected such that $N^{-p/2} \longrightarrow 0$ as $N \longrightarrow \infty\ \forall p \geq 2$. By applying the polynomial identity principle it then follows that

$$\begin{cases} (i) & \dot{m} = \zeta m - (\zeta + \iota)m^2 \\ (ii) & \partial_t Q(s) = [2(\zeta + \iota)m - \zeta]\partial_s(sQ(s)) + [\zeta m(1-m) + \iota m^2]\frac{1}{2}\partial_s^2 Q(s) \end{cases} \tag{A.17}$$

where the macroscopic equation (A.17–*i*) gives a logistic dynamics. The non linearity is due to the rate functions (A.15) and (A.16) which account for external field effects on transition rates (A.4) and (A.5).

The macroscopic equation (A.17–*i*) is an ODE, hence with an initial condition $m(0) = m_0 = N_1(0) = N$ it allows for a logistic dynamics with multiple equilibria: $\in \{0, \zeta/(\zeta + \iota)\}$. The stable equilibrium is $m^* = \zeta/(\zeta + \iota)$ and the general solution is

$$m(t) = \frac{m_0 \zeta e^{\zeta t}}{\zeta + m_0(\zeta + \iota)(e^{\zeta t} - 1)} \tag{A.18}$$

which describes the evolution of $\langle N^1(t)\rangle = Nm(t)$ as the expected, *i.e.* most probable, drifting path.

Appendix B

Herein a solution for the Fokker-Planck equation (A.17–*ii*) is found in terms of *Q*(*s*). Using $\overline{Q}(s)$ to indicate the stationary probability for , by setting $\dot{Q} = 0$ it follows that

$$[2(\zeta + \imath)m^* - \zeta](s\overline{Q}(s)) = \frac{\zeta m^*(1-m^*)+\imath m^{*2}}{2}\partial_s^2\overline{Q}(s)$$

By direct integration it gives

$$\overline{Q}(s) = C\exp\left[\frac{\zeta - 2(\zeta+\imath)m^*}{\zeta m^* + (\imath - \zeta)m^{*2}}s^2\right] \tag{B.1}$$

By substituting for $m^* = \zeta / \zeta + \imath)$ it can be found that

$$\overline{Q}(s) = C\exp\left(-\frac{s^2}{2\sigma^2}\right) : \sigma^2 = \frac{\zeta\imath}{(\zeta+\imath)^2} \tag{B.2}$$

being $C = 1/\int \exp\left(-\frac{s^2}{2\sigma^2}\right)ds$ a normalization constant.

Since the variance is a function of ζ and ι, which are time dependent, this representation allows for *stochastic determinism*. That is, the stationary solution for the distribution of spreading fluctuations still performs some *vibrating* volatility due to the exchange of agents between the two states. These exchanges let the volumes almost constant on expectation through time when approaching the stable equilibrium, but fluctuations depend on *who* is jumping because agents jump from one state to another carrying their own characteristics and endowments. Unfortunately these individual jumps are unobservable, and agents are indistinguishable, from a macroscopic point of view. Nevertheless, it known it happens and this let macroscopic observables to *vibrate* about some equilibrium path. On the other hand, equilibrium itself is a state of nature for the system as a whole, it is not a property of its elementary constituents; equilibrium is a probability distribution for agents over a space of states and not a point of balance of two forces.

Appendix C

The basic steps for deriving of the steady state probability are here sketched, referring the interested reader to the cited references. Stationary probability can be obtained by applying Brook's lemma (Brook, 1964) which defines local characteristic of continuous Markov chains. Hammersley and Clifford demonstrate that, under opportune conditions, for each Markov random field there is one and only one Gibbs random field, and define the functional form for the conjunct probability structure once the neighbourhood relations have been identified (Clifford, 1990). The expected stationary probability (40) of the Markovian process for N^1, when detailed balance holds, can be expressed by:

$$P^e(N_k) \propto Z^{-1} e^{-\beta N U(N_k)} \tag{C.1}$$

where $U(x)$ is the Gibbs potential and can be defined as a functional of the local dynamic characteristics of the state variable N_k. In particular:

$$e^{\beta g(N^1)} + e^{-\beta g(N^1)} = N \tag{C.2}$$

The above formulation leads (Aoki, 2002) to an explicit formulation for the probability η as a function of the state variable N^1:

$$\eta(N^1) = N^{-1} e^{\beta g(N^1)} \tag{C.3}$$

where $g(N^1)$ is a function that evaluates the relative difference in the outcome as a function of N^1. β may be interpreted as an inverse measure of the system uncertainty. The uncertainty among the different possible configurations in a stochastic system can be evaluated through a statistical entropy measure (Balian, 1991). The quantification of the parameter β can be obtained by maximising the statistical entropy of the system (Jaynes, 1957). In the present case the problem is configured as follows:

$$\begin{cases} \max H(N^1, N^0) = -N^1(t)\ln(N^1(t)) - N^0(t)\ln(N^0(t)) \quad s.t. \\ N^1(t) + N^0(t) = N \\ N^1(t)y^1(t) + N^0(t)y^0(t) = Y(t) \\ \end{cases} \tag{C.4}$$

The first of the two constraints ensures the normalization of the probability function. The second ensures that all the wealth in the system is generated by firms in the two kind of states. The solution of the maximisation problem (for details see Di Guilmi, 2008) yields

$$\beta(t) = \ln\left(-\frac{y^1(t) - \bar{y}(t)}{y^0(t) - \bar{y}(t)} \right)\left(y^1(t) - y^0(t)\right)^{-1} \tag{C.5}$$

Large values of β associated with positive values of $g(N^1)$ cause $\eta(N^1)$ to be larger than $1 - \eta(N^1)$, making the transition from state 0 to state 1 more likely to occur than the opposite one. In binary models and for great N, the equation of the potential is:

$$U(N^j) = -2\int_0^{N^j} g(z)dz - \frac{1}{\beta} H(\underline{N})$$

where $H(\underline{N})$ is the Shannon entropy with $\underline{N} = (N^1, N^0)$. In order to find the stationary points of probability dynamics we need to individuate its peak (if it exists). β is an inverse multiplicative factor for entropy: a relative high value of β means that the uncertainty in the system is low, with few firms exposed at bankruptcy risk. For values of β around 0, and a more relevant volatility in the system, in order to find the peak of probability dynamics we need to find the local minimum of the potential. Aoki (2002) shows that the points in which the potential is minimized are also the critical point of the aggregate dynamics of $P^e(N_k)$. Deriving the potential with respect to N^1 and then setting $U' = 0$:

$$g(N^1) = -\frac{1}{2\beta}\frac{dH}{dN^1} = -\frac{1}{2\beta}\ln\left(\frac{N^1}{N - N^1}\right) \tag{C.6}$$

and using equation (C.5),an explicit formulation for $g(N^1)$ is found in stationary conditions:

$$g(N^1) = \frac{y^0 - y^1}{2}$$

that quantifies the mean difference (for states) of the output.