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A MODEL OF THE STOCHASTIC CONVERGENCE BETWEEN BUSINESS CYCLES

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Abstract

A new non-linear parametric model, the Stochastic Cyclical Convergence Model (SCCM), is proposed for measuring the convergence dynamics between two cycles. It combines unobserved component models with time-varying parameter models. The convergence between the two cycles is characterised by two time-varying parameters, the phase-shift and the phase-adjusted correlation. A Kalman smoother-based iterative procedure is developed for the model estimation. The procedure is checked on simulated time series and the biases are quantified. SCCM models are applied to UK, French, German and Euro-zone business cycles over the period 1960:1-2003:1. The result is twofold: on the one hand, the British cycle progressively synchronises with that of the Euro-zone, with a phase-shift decreasing from approximately 5 quarters in 1990:1 to 2 quarters in 2003:1. On the other hand, the phase-adjusted correlation between the UK and Euro-zone cycles falls from approximately 90% in 1990:1 to 60% in 2003:1. Moreover, the French cycle converges more towards the German cycle, than the UK one. UK shocks have asynchronous asymmetric effects and the adoption of the euro would require improved fiscal coordination.

Keywords: convergence, synchronisation, business cycles, multivariate unobserved components models, time-varying parameter models, Kalman filter.

JEL *Classification*: C13, C32, E32.

Résumé

Cet article propose un nouveau modèle paramétrique non-linéaire, dénommé le modèle de convergence cyclique stochastique (SCCM), qui permet de mesurer les dynamiques de convergence entre deux cycles. Ce modèle combine des modèles à composantes inobservables avec des modèles à paramètres évolutifs. La convergence entre deux cycles est caractérisée par deux paramètres évolutifs, la phase et la corrélation corrigée de la phase. Une procédure itérative fondée sur le lisseur de Kalman est développée pour estimer le modèle. Cette procédure est évaluée sur des données simulées et les biais des estimateurs sont quantifis. Des modèles SCCM sont estimés pour les cycles anglais, français, allemand et de la zone euro, sur la période 1960:1-2003:1. Le principal résultat de cette méthode a deux dimensions: d'une part, le cycle anglais se synchronise progressivement avec celui de la zone euro, puisque la phase passe de environ cinq trimestres en 1990:1 à deux trimestres en 2003:1. D'autre part, la corrélation, corrigée de la phase, entre les cycles du Royaume-Uni et de la zone Euro tombe d'environ 90% en 1990:1 à 60% en 2003:1. De plus, le cycle français converge plus que le cycle anglais vers le cycle allemand. Les chocs économiques auraient ainsi encore des effets asynchrones asymétriques au Royaume-Uni et l'adoption de l'euro nécessiterait des progrès en terme de coordination budgétaire.

Mots-clefs: convergence, synchronisation, cycles d'affaire, modèles à composantes inobservables multivariés, modèles à paramètres évolutifs, filtre de Kalman.

Codes **JEL** : C13, C32, E32.

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1 Introduction

Since the seminal paper on the Optimum Currency Area Theory (Mundell, 1961), the degree of cyclical convergence among a group of countries has become a key criterion to assess whether a currency area is feasible. The evaluation of this criterion influences the overall organisation of a currency area. In particular, a lack of cyclical convergence might increase the role of fiscal coordination.

Many approaches are possible for analysing the convergence between cycles. For example, to assess whether UK and Euro-zone cycles have converged, the HM Treasury (2003) takes into account four different quantities: the closeness of the main macroeconomic variables in the recent period; the degree of cyclical convergence during the past decades, to measure the risk of cyclical divergence in the future (Artis and Zhang, 1997); some structural factors like the structure of the trade balance, the oil production asymmetry or housing markets differences; some endogenous factors like the trade effects associated with integration into a currency area (Rose, 1999).

Empirical studies simply based on correlation coefficients (e.g. Artis and Zhang, 1997, Angeloni and Dedola, 1999 and Wynne and Koo, 2000) have generally established the existence of an increased convergence between euro-zone business cycles during the Exchange Currency Mechanism (ECM) period. Belo (2001) has confirmed the existence of convergence within the euro area using annual data from 1960 to 1999. He has also shown a persistent lead of the UK cycle over that of the Euro-zone and a strong association between the two cycles, after correcting for this lead. Following the approach of the historical convergence of business cycles, we try to improve on sub-sample correlation analysis, by modelling the convergence dynamics with time-varying parameters (see below).

Before addressing the convergence modelling issue, the business cycle has to be defined. *Classical business cycles* were initially defined by Burns and Mitchell (1946) as a comovement among economic time series in terms of level (employment, business surveys, etc.). These have been the subject of some recent papers (Harding and Pagan, 1999, Artis, Krolzig, and Toro, 2002), that focus on turning-point analysis, in which extended Markow Switching Vector Auto-Regressive (MS-VAR) models are applied (Hamilton, 1989). But the business cycles have received a new definition (Zarnowitz, 1985) as being deviations from long-run trends rather than levels of economic aggregates and they are called in this context *growth cycles*. This approach makes it possible to distinguish transitory shocks from permanent ones, a distinction that is useful for the coordination of economic policies. The growth cycle definition is adopted in this paper, because the identification of transitory shocks is necessary for analysing their symmetry in a currency area.

Many growth cycle models are available. In this paper, growth cycles are modelled with multivariate unobserved components models even if, for reasons of parsimony, they are pre-estimated with the band-pass Hodrick-Prescott (HP) filter. Other growth cycle models, like Beveridge-Nelson decomposition or Structural Vector Auto-Regressive models (SVAR), are not considered, since they provide uninterpretable results: the transitory components do not show enough persistence, a property that is generally expected from a cycle, and are sometimes negatively correlated with the capacity utilization rate, which is assumed to be closely related to the business cycle (Camba-Mendez and Rodriguez-Palenzuela, 2003).

As shown by Belo (2001), the phase-shift among business cycles of different countries might change the diagnosis concerning the correlation. Rünstler (2004) extends the multivariate unobserved component model, in order to take into account such phase-shifts and to measure phase-adjusted correlation coefficients. This extension is presented in Section 2.2 and will be used in this paper.

The association degree between business cycles in a group of countries has first been defined by the presence of common components in Vector Auto-Regressive (VAR) models. Cofeature tests proposed by Engle and Kozicki (1993) and Vahid and Engle (1993) have been applied, for example, by Mills and Holmes (1999) to European countries over different exchange-rate-regime time periods. Because such an association criterion is very restrictive, Kose, Prasad, and Terrones (2003), Stock and Watson (2003) and Bordo and Helbling (2003) have preferred to use factor models (initially developped by Stock and Watson, 1991) and their extended versions for measuring the share of variance explained by a common factor. As this seems to be an interesting way to formulate an association indicator in a multivariate model, the Rünstler model is reformulated here in a common factor framework (Section 2.3).

Cyclical convergence is generally studied with *ad hoc* indicators (for example correlation coefficients in Artis and Zhang, 1997) applied to estimated cycles on various sub-samples. As far as we know, few models have been developed for studying the cyclical convergence dynamics. Recently, Koopman and Azevedo (2004) have proposed a multivariate unobserved components model that tries to incorporate these dynamics: the convergence is associated with a progressive reduction of the phase-shift and an increase in the correlation between the two cycles. Koopman and Azevedo (2004) have modelled convergence dynamics using logistic functions, by extending a model initially developed by Rünstler (2004). But logistic functions are monotonous and do not allow transitory divergences, thus they do not allow convergence and divergence movements to occur successively. Indeed, each turning point is an opportunity for divergence, for example some countries might have begun their recoveries, while others remain in deep recession.

Such recurrent divergences require a stochastic model of phase-shift and correlation coefficients, for example a random walk model. Models with stochastic variation of regression parameters were first proposed in Cooley and Prescott (1976). In the unobserved component framework, a stochastic covariance model is proposed by Harvey, Ruiz, and Shepard (1994). Boone (1997) uses time-varying parameter (TVP) regressions for measuring convergence of supply and demand shocks, in a SVAR model. Engle (2002) formalises this mechanism in a multivariate Generalised Auto-Regressive Conditional Heteroskedastic (GARCH) model of dynamic conditional correlation. Although they do not take into account any phase-shift, Hallett and Richter (2004) propose an interesting approach of the convergence between US and European business cycles: they use TVP models for studying the evolution of the coherence (a frequency-domain concept).

In this paper, a new bivariate model is proposed, the *Stochastic Cyclical Convergence Model* (SCCM), in which the phase-shift and the correlation between the two cycles follow random walk dynamics. Contrary to the logistic function specification of Koopman and Azevedo (2004), the random walk processes allow for successive convergence and divergence movements. As this model is non-linear, a specific procedure, the *Iterative Kalman Smoother* (IKS), is also proposed for its estimation.

The next section contains a description of the unobserved component models of the phase-shifted cycles, the SCCM model and its estimation procedure. In Section 3, the estimation procedure is checked on simulated data. In Section 4, SCCM models are applied to UK and Euro-zone cycles from 1960:1 to 2003:1, in order to try to verify the first test proposed by the Chancellor of the Exchequer for UK entry into the Euro-zone. Finally, in the same section, this analysis of the UK convergence will be supplemented by a comparison with the French convergence towards the German business cycle.

2 A model of stochastic cyclical convergence

In this section, the bivariate stochastic cycle model (Section 2.1), its extension with phase-shifts (Section 2.2) and its reformulation in a common factor framework (Section 2.3) are reviewed before introducing the SCCM model (Section 2.4).

2.1 Bivariate stochastic cycle model

To model cyclical dynamics of a stationary time series vector $\mathbf{y}_t$, we first consider the *multivariate stochastic cycle model* developed by Harvey and Koopman (1997). Such models are called *Seemingly Unrelated Time Series Equations* (Harvey, 1989).

The bivariate case will be sufficient for the analysis carried on in this article. $\mathbf{y}_t = [y_{1,t}, y_{2,t}]'$ is a vector of observations that depends on two unobserved components that are also vectors: a cycle $\boldsymbol{\psi}_t = [\psi_{1,t}, \psi_{2,t}]'$, and an irregular term $\boldsymbol{\varepsilon}_t = [\varepsilon_{1,t}, \varepsilon_{2,t}]'$. Measurement and state equations are written as follows:

$$\begin{cases} y_{i,t} = [1,0]\bar{\boldsymbol{\psi}}_{i,t} + \varepsilon_{i,t} \\ \bar{\boldsymbol{\psi}}_{i,t} = \phi T_\lambda \bar{\boldsymbol{\psi}}_{i,t-1} + \bar{\boldsymbol{\kappa}}_{i,t} \end{cases} \tag{1}$$

with $i = 1, 2$, $t = 1, ..., n$, $\varepsilon_{i,t}$ are white independent Gaussian noises with standard errors $\sigma_{\varepsilon,i}$,

$$\bar{\boldsymbol{\psi}}_{i,t} = \begin{bmatrix} \psi_{i,t} \\ \psi^+_{i,t} \end{bmatrix}, \bar{\boldsymbol{\kappa}}_{i,t} = \begin{bmatrix} \kappa_{i,t} \\ \kappa^+_{i,t} \end{bmatrix} \text{ and } T_\lambda = \begin{bmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{bmatrix}.$$

Both measurement equations in the first part of the system (1) are the decomposition of series into cycle and irregular components. In the second part of the system (1), the state equations have a similar iterative form for both time series. Parameters λ and ϕ are the frequency and the damping factor of both cycles. The disturbance vectors $\boldsymbol{\kappa}_t = [\kappa_{1,t}, \kappa_{2,t}]'$ and $\boldsymbol{\kappa}^+_t = [\kappa^+_{1,t}, \kappa^+_{2,t}]'$ are bivariate normal disturbances, which are mutually uncorrelated at all time periods and have covariance matrices Σ_κ:

$$\begin{bmatrix} \kappa_{1,t} \\ \kappa_{2,t} \end{bmatrix} \sim NID(0, \Sigma_\kappa), \begin{bmatrix} \kappa_{1,t}^{+} \\ \kappa_{2,t}^{+} \end{bmatrix} \sim NID(0, \Sigma_\kappa) \text{ and } \Sigma_\kappa = \begin{bmatrix} \sigma_{\kappa,1}^2 & \rho\sigma_{\kappa,1}\sigma_{\kappa,2} \\ \rho\sigma_{\kappa,1}\sigma_{\kappa,2} & \sigma_{\kappa,2}^2 \end{bmatrix} \tag{2}$$

where the elements of the covariance matrix (Σ_κ) are defined via the standard errors ($\sigma_{\kappa,1}$, $\sigma_{\kappa,2}$) and the correlation (ρ) of the cycle innovations ($\kappa_{1,t}$ and $\kappa_{2,t}$). One can then verify that the cycles are stationary ARMA(2, 1) processes (when $\phi < 1$) and

$$\Gamma(\tau) = (1 - \phi^2)^{-1}\phi^{|\tau|}\cos(\lambda\tau)\Sigma_\kappa \tag{3}$$

is the cross-covariance function. The cross-covariance functions show that the covariance between the two cycles is maximised when cycles are not shifted ($\tau = 0$). It is assumed that $0 \le \phi < 1$ and $0 \le \lambda \le \pi$, which implies that the cycles $\psi_{i,t}$ are stationary cycles. The roots of the autoregressive part are constrained to lie in the complex region, which is the usual condition of cyclical behaviour. When $\lambda = 0$ or $\lambda = \pi$, the cycle $\psi_{i,t}$ degenerates into a AR(1) process. If the damping factor ϕ takes the value 0, the cycle $\psi_{i,t}$ degenerates into a white noise.

The damping factor ϕ and the frequency λ are the same in both series. Cycles with the same coefficients ϕ and λ are called 'similar cycles'. The shapes of such cycles can be different, because they depend on the variance of the disturbance, which is specific to each series. Similar cycles are not necessarily synchronised among series, because tthe sequence of the disturbances is peculiar to each series. When cycles are perfectly synchronised ($\psi_{2t} = a\psi_{1t}$), cycles are called 'common cycles'. The only difference is in their strength: each cycle is proportional to the others. Common cycles imply much stronger restrictions than similar cycles.

2.2 Bivariate stochastic cycle model with phase-shifts

The model developed by Harvey and Koopman (1997) has been extended by Rünstler (2004), who developed a *multivariate stochastic cycle model with phase-shifts*. In the bivariate case, measurement and state equations become

$$\begin{cases} y_{i,t} = [\cos(\lambda\xi_i), \sin(\lambda\xi_i)]\, \bar{\boldsymbol{\psi}}_{i,t} + \varepsilon_{i,t} \\ \bar{\boldsymbol{\psi}}_{i,t} = \phi T_\lambda \bar{\boldsymbol{\psi}}_{i,t-1} + \bar{\boldsymbol{\kappa}}_{i,t} \end{cases} \tag{4}$$

The parameter ξ_i is the shift between both cycles. The normalisation $\xi_1 = 0$ is imposed. The first equation is the decomposition of the first series into cycle and irregular components. The second equation has the same structure, but the cyclical component is expressed as a function of hidden cycles $\psi_{2,t}$ and $\psi_{2,t}^{+}$, since $\xi_2 \neq 0$. As shown below, $\psi_{1,t}$ and $\psi_{2,t}$ are synchronised, but the transformation of $\psi_{2,t}$ and $\psi_{2,t}^{+}$ creates a "phase-shift" equal to ξ_2 periods between the second cycle and the first one. The correlation ρ between cyclical innovations is called the phase-ajusted correlation and the contemporaneous correlation between both cycles is equal to $\rho\cos(\lambda\xi_2)$. The constraints $\rho \ge 0$ and $-\pi < \lambda\xi_2 < \pi$ are assumed[1]. The cycles are stationary processes with the

[1]Using basic trigonometric identities, it can be shown that such constraints are equivalent to the constraint $-\pi/2 < \lambda\xi_2 < \pi/2$ assumed in Rünstler (2004).

following cross-covariance function[2]:

$$\Gamma(\tau) = \frac{\phi^{|\tau|}}{1-\phi^2}\begin{bmatrix} \sigma^2_{\kappa,1}\cos(\lambda\tau) & \rho\sigma_{\kappa,1}\sigma_{\kappa,2}\cos\left[\lambda(\tau+\xi_2)\right] \\ \rho\sigma_{\kappa,1}\sigma_{\kappa,2}\cos\left[\lambda(\tau-\xi_2)\right] & \sigma^2_{\kappa,2}\cos(\lambda\tau) \end{bmatrix} \tag{5}$$

If the identification conditions ($0 < \phi < 1, 0 < \lambda < \pi, 0 < \rho$) are assumed, the cross-covariance between the cycles is maximised when the lag operator is applied $floor(\xi_2)$ times on the second cycle[3]. If $\phi = 0$, $\lambda = 0$, $\lambda = \pi$ or $\rho = 0$, the cross-covariance does not depend on ξ_2 and the phase-shift ξ_2 is not identifiable.

2.3 Reformulation in a single common factor model

It is interesting to reformulate the Rünstler model in a *single common factor model*[4] (both models are statistically equivalent, see Koopman and Azevedo (2004)). Such a formulation allows to distinguish common from idiosyncratic shocks. Such a model is written as follows,

$$\begin{cases} y_{i,t} = a_i\left[\cos(\lambda\xi_i),\sin(\lambda\xi_i)\right]\bar{\boldsymbol{\psi}}^c_t + [1,0]\bar{\boldsymbol{\psi}}^*_{i,t} + \varepsilon_{i,t} \\ \bar{\boldsymbol{\psi}}^c_t = \phi T_\lambda \bar{\boldsymbol{\psi}}^c_{t-1} + \bar{\boldsymbol{\kappa}}^c_t \\ \bar{\boldsymbol{\psi}}^*_{i,t} = \phi T_\lambda \bar{\boldsymbol{\psi}}^*_{i,t-1} + \bar{\boldsymbol{\kappa}}^*_{i,t} \end{cases} \tag{6}$$

Each series $y_{i,t}$ can be decomposed into a common component (a phase-shift transformation of the common cycle ψ^c_t), a specific cycle $\psi^*_{i,t}$ and an irregular component $\varepsilon_{i,t}$. The parameter a_i is the weight of the common cycle in each series. It is assumed that $-\pi < \lambda\xi_2 < \pi$ and $a_i \geq 0$. All innovation variables $(\kappa^c_{i,t}, \kappa^{+c}_{i,t}, \kappa^*_{i,t}, \kappa^{+*}_{i,t})$ are assumed to be Gaussian, mutually and serially uncorrelated at all time points and to have variances $(\sigma^2_{\kappa,c}, \sigma^2_{\kappa,c}, h^2_i, h^2_i)$.

By considering disturbances, the previous decomposition of each global cycle as the sum of the common and specific cycles can be written as the decomposition of global disturbances κ_t into weighted sums of common and idiosyncratic disturbances (κ^c_t and κ^*_t):

$$\boldsymbol{\kappa}_t = \left[\mathbf{a}\odot\cos(\lambda\boldsymbol{\xi})\right]\boldsymbol{\kappa}^c_t + \left[\mathbf{a}\odot\sin(\lambda\boldsymbol{\xi})\right]\boldsymbol{\kappa}^{+c}_t + \boldsymbol{\kappa}^*_t$$

where

$$\begin{aligned} \left[\mathbf{a}\odot\cos(\lambda\boldsymbol{\xi})\right] &= \left[a_1, a_2\cos(\lambda\xi_2)\right]', \\ \left[\mathbf{a}\odot\sin(\lambda\boldsymbol{\xi})\right] &= \left[0, a_2\sin(\lambda\xi_2)\right]', \end{aligned}$$

with $\boldsymbol{\kappa}_t = [\kappa_{1,t},\kappa_{2,t}]' \sim N(0,\Sigma_\kappa)$, $\tilde{\boldsymbol{\kappa}}^c_t = \left[\kappa^c_t, \kappa^{+c}_t\right]' \sim N(0, \sigma^2_{\kappa,c}I_2)$ and $\boldsymbol{\kappa}^*_t = \left[\kappa^*_{1,t},\kappa^*_{2,t}\right]' \sim N(0,H_\kappa)$. $\mathbf{a}$ is the vector of weight coefficients, $\boldsymbol{\xi}$ is the vector of shift coefficients and H_κ is the diagonal matrix of specific disturbance variances. The symbol $\odot$ indicates the coefficient-by-coefficient product, which can be used for vectors or matrices with equal dimensions. The disturbance decomposition is unique if and only if the global disturbance variance Σ_κ has the following unique decomposition

$$\Sigma_\kappa = \sigma^2_{\kappa,c}aa'\odot\Lambda + H_\kappa, \text{ with } \Lambda = \begin{bmatrix} 1 & \cos(\lambda\xi_2) \\ \cos(\lambda\xi_2) & 1 \end{bmatrix}.$$

[2]Proof of the auto-covariance expression is given by Rünstler (2004).

[3]For a real x, the $floor$ function returns the largest integer not greater than x.

[4]Factor models were initially used in a SVAR framework (Stock and Watson, 1991).

This implies restrictions on the vector $\mathbf{a}$ and on the matrix H_κ. Thus, besides $\xi_1 = 0$, the normalisation constraints $a_1 = 1$ and $h_1 = 0$ are also imposed. As the matrix Σ_κ has three degrees of freedom, the three parameters $\sigma^2_{\kappa,c}$, a_2 and h_2 are identified. Because of the restrictions, the first global disturbance is equal to the common disturbance: $\kappa_{1,t} = \kappa^c_t$,$\kappa^+_{1,t} = \kappa^{+c}_t$. Thus, the two series play different roles in the model: conventionally, a *cycle of interest* is distinguished from a *reference cycle*. The properties of the cycle of interest are studied in comparison with the reference cycle. The relation between the two cycles is characterised by two main properties: its phase and its weight. The phase-shift is the lag relative to the reference cycle. The weight is a sort of amplitude ratio of the cycle of interest in comparison with the reference cycle. The relation between cycles can equivalently be characterised by the phase-shift and the phase-adjusted correlation ρ, which can be computed with the following formula[5]:

$$\rho = a_2 / \sqrt{a_2^2 + h_2^2/\sigma^2_{\kappa,c}}. \tag{7}$$

It is easily verified that $0 \leqslant \rho \leqslant 1$ is always true. The phase-shift identification condition ($\rho > 0$) is equivalent to ($\sigma_{\kappa,c} > 0, a_2 > 0$). As explained in Rünstler (2004), the phase-shift and the phase-adjusted correlation concepts are closely related to frequency-domain concepts (the phase and the coherence). But the model-based approach avoid potential distorsions of the filter-based approach, as documented by Harvey and Trimbur (2003).

If there is no specific cycle ($h_2 = 0$), the correlation between the two phase-adjusted cycles is equal to one ($\rho = 1$). In other words, the rank of the disturbance covariance matrix is equal to one. In a sense closed to the initial definition of Engle and Kozicki (1993), both series share a common cycle. In this model, the common cycle case is less restrictive than in the Harvey and Koopman (1997) model: cycles are not completely synchronised (they can be phase-shifted in relation to each other).

2.4 Stochastic Cyclical Convergence Model (SCCM)

We extend the previous bivariate model (i.e. a multivariate stochastic cycle model with phase-shifts expressed in a single common factor framework) to incorporate the convergence dynamics. This extension differs from that of Koopman and Azevedo (2004). Each property of the cycle of interest, regard to the reference cycle, is modelled with a time-varying (tv) parameter: the convergence is characterised by the *tv-phase* $\xi_{2,t}$ and the *tv-weight* $a_{2,t}$. These tv-parameters are random walk processes. The normalisation conditions become: $\xi_{1,t} = 0$ and $a_{1,t} = 1$. The Stochastic Cyclical Convergence Model (SCCM) is written as follows:

$$\begin{aligned}
&y_{i,t} = a_{i,t}\left[\cos(\lambda\xi_{i,t}), \sin(\lambda\xi_{i,t})\right]\bar{\boldsymbol{\psi}}^c_t + [1,0]\,\bar{\boldsymbol{\psi}}^*_{i,t} + \varepsilon_{i,t}\\
&\bar{\boldsymbol{\psi}}^c_t = \phi T_\lambda \bar{\boldsymbol{\psi}}^c_{t-1} + \bar{\boldsymbol{\kappa}}^c_t\\
&\bar{\boldsymbol{\psi}}^*_{i,t} = \phi T_\lambda \bar{\boldsymbol{\psi}}^*_{i,t-1} + \bar{\boldsymbol{\kappa}}^*_{i,t}\\
&a_{i,t} = a_{i,t-1} + \gamma_{i,t}\\
&\xi_{i,t} = \xi_{i,t-1} + \delta_{i,t}
\end{aligned} \tag{8}$$

[5]The contemporaneous correlation $Corr(\psi_{1,t}, \psi_{2,t})$ between both cycles would be $\cos(\lambda\xi_2)\rho$.

$\varepsilon_{i,t}$, κ_t^c, κ_t^{+c}, $\kappa_{i,t}^*$, $\kappa_{i,t}^{+*}$, $\gamma_{i,t}$ and $\delta_{i,t}$ are white independent Gaussian noises, with standard errors $\sigma_{\varepsilon,i}, \sigma_{\kappa,c}, \sigma_{\kappa,c}, h_i, h_i, \sigma_{\gamma,i}$ and $\sigma_{\delta,i}$. Using the normalisation constraints,

$$a_{1,t} = 1, \xi_{1,t} = 0, h_1 = 0, \tag{9}$$

in this particular formulation, the second cycle is generated by the propagation of the first cycle. The propagation is characterised by the tv-phase ($\xi_{2,t}$) and the tv-weight ($a_{2,t}$), which should again verify $-\pi < \lambda\xi_{2,t} < \pi$ and $a_{2,t} \geq 0$. But it can equivalently be characterised by the tv-phase ($\xi_{2,t}$) and the phase-adjusted tv-correlation (ρ_t), by computing:

$$\rho_t = a_{2,t} / \sqrt{a_{2,t}^2 + h_2^2 / \sigma_{\kappa,c}^2}$$

In this case, the auto-covariance function[6] of the cycles at a lag $\tau > 0$ and a date t, conditional on a realisation of the tv-correlation time series $\boldsymbol{\rho}_{1:t} = [\rho_{2,1}, ..., \rho_{2,t}]'$ and of the tv-phase one $\boldsymbol{\xi}_{1:n} = [\xi_{2,1}, ..., \xi_{2,n}]'$ is:

$$\Gamma_{a,\xi}(\tau; t) = \frac{\phi^\tau}{1-\phi^2} \begin{bmatrix} \sigma_{\kappa,1}^2 \cos(\lambda\tau) & \tilde{\rho}_{t-\tau}\sigma_{\kappa,1}\sigma_{\kappa,2} \cos\left[\lambda(\tau + \xi_{2,t})\right] \\ \tilde{\rho}_{t-\tau}\sigma_{\kappa,1}\sigma_{\kappa,2} \cos\left[\lambda(\tau - \xi_{2,t-\tau})\right] & \sigma_{\kappa,2}^2 \cos(\lambda\tau) \end{bmatrix}, \tag{10}$$

with $\tilde{\rho}_t = (1-\phi^2)\sum_{i=0}^{t} \phi^{2i}\rho_{t-i}$. At a lag $\tau < 0$, the autocovariance is given by

$$\Gamma_{\rho,\xi}(\tau, t) = \frac{\phi^{-\tau}}{1-\phi^2} \begin{bmatrix} \sigma_{\kappa,1}^2 \cos(\lambda\tau) & \tilde{\rho}_t\sigma_{\kappa,1}\sigma_{\kappa,2} \cos\left[\lambda(\tau + \xi_{2,t})\right] \\ \tilde{\rho}_t\sigma_{\kappa,1}\sigma_{\kappa,2} \cos\left[\lambda(\tau - \xi_{2,t-\tau})\right] & \sigma_{\kappa,2}^2 \cos(\lambda\tau) \end{bmatrix}. \tag{11}$$

At time t, if $\phi = 0$, $\lambda = 0$, $\lambda = \pi$ or ($\boldsymbol{\rho}_{1:t} = \mathbf{0}_t$), the autocovariance matrix $\Gamma_{\rho,\xi}(\tau, t)$ does not depend on $\xi_{2,t}$ and the phase-shift $\xi_{2,t}$ is not identifiable. Thus, the identification conditions $\phi > 0$, $0 < \lambda < \pi$ and ($\rho_t > 0$, $\forall t$) are assumed.

The cyclical convergence at a date t is defined in this paper by the two following conditions: perfect synchronisation ($\xi_{2,t} = 0$) and perfect correlation ($\rho_t = 1$). In the case of a perfect and permanent synchronisation, the phase-shift is assumed constant ($\sigma_{\delta,2} = 0$) and equal to zero ($\xi_{2,0} = 0$). For the perfect symmetric case, the correlation is assumed constant and equal to one, which is equivalent to assuming ($\sigma_{\gamma,2} = 0, h_2 = 0$). Finally, if $\sigma_{\delta,2} = 0$, $\xi_{2,0} = 0$, $\sigma_{\gamma,2} = 0$ and $h_2 = 0$ simultaneously, cycles are synchronised and symmetric. This is the common cycle case, initially defined by Engle and Kozicki (1993): a linear combination of the cycles is a white noise; the only difference between the two cycles is their amplitude.

2.5 Estimation methodology

As the issue raised does not directly concern the trend/cycle decomposition, if series of interest $x_{i,t}$ and $x_{2,t}$ have a trend, they are detrended and smoothed with band-pass HP filters before the iterative estimation procedure:

$$y_{i,t} = HP_\alpha(x_{i,t} - HP_\beta(x_{i,t})) \tag{12}$$

[6]Proof of the auto-covariance expression is given in Appendix A.

where the operator HP transforms a series x into its trend, using a Hodrick-Prescott filter. The choice of the parameter of the operator (here α or β) depends on the considered series. For the the empirical application in section 4, where series of interest are quarterly GDP time series, periods between 1.5 and 10 years are filtered[7] by setting $\alpha = 1$ and $\beta = 1,600$.

The parameter vector of the SCCM model, denoted by $\mathbf{p} = \left[\phi, \lambda, h_2, \sigma_{\kappa,c}, \xi_{2,0}, \sigma_{\delta,2}, \theta_{2,0}, \sigma_{\gamma,2}\right]'$ for convenience[8] contains: the damping factor (ϕ), the frequency (λ), the specific and common standard deviations ($h_2, \sigma_{\kappa,c}$), the tv-shift parameters ($\xi_{2,0}, \sigma_{\delta,2}$) and the tv-weight parameters ($\theta_{2,0}, \sigma_{\gamma,2}$). The tv-parameters ($\xi_{2,t}, a_{2,t}$) and $\mathbf{p}$ are transformed, in order to facilitate their estimation and to take into account constraints. The transformed parameters and tv-parameters are called $\boldsymbol{\theta}_{\mathbf{p}} = [\theta_\phi, \theta_\lambda, \theta_h, \theta_{\kappa,c}, \theta_{\xi,0}, \theta_\delta, \theta_{a,0}, \theta_\gamma]'$ and $(\theta_{\xi,t}, \theta_{a,t})$. The parameters can then be calculated by the following formula:

$$\begin{aligned} \phi &= \frac{|\theta_\phi|}{\sqrt{1+\theta_\phi^2}}, \lambda = \frac{2\pi}{2+|\theta_\lambda|}, h_2 = \theta_h, \sigma_{\kappa,c} = \theta_{\kappa,c}, \\ \xi_{2,0} &= \frac{2+|\theta_\lambda|}{2\pi}\theta_{\xi,0}, \sigma_{\delta,2} = \frac{2+|\theta_\lambda|}{2\pi}\frac{\theta_\delta}{\sqrt{n}}, a_{2,0} = \theta_{a,0}, \sigma_{\gamma,2} = \frac{\theta_\gamma}{\sqrt{n}}. \end{aligned}$$

and the tv-parameters by

$$\xi_{2,t} = \frac{2+|\theta_\lambda|}{2\pi}\theta_{\xi,t}, a_{2,t} = \theta_{2,t}.$$

The damping factor (ϕ) is not allowed to be negative or higher than 1. The tv-phase parameters ($\xi_{2,0}$ and σ_δ) are rescaled with the frequency λ. This transformation imply a simpler constraint $-\pi < \theta_{\xi,t} < \pi$, instead of $-\pi/\lambda < \xi_{2,t} < \pi/\lambda$. The standard deviations $\sigma_{\delta,2}$ and $\sigma_{\gamma,2}$ are rescaled with the square root of the sample size ($\sqrt{n}$). This transformation, used in Stock and Watson (1998), allows the direct definition of the order of magnitude of $\theta_{\xi,t}$ and $\theta_{a,t}$ (instead of their innovations) by the parameters θ_δ and θ_γ .

The difficulty in estimating the SCCM model with the filtered series $y_{1,t}$ and $y_{2,t}$, then, comes from the non-linear transformation of the tv-parameters $\theta_{\xi,t}$ and $\theta_{a,t}$ in the measure equation. The model cannot be written in a linear state-space form and a linear Kalman filter cannot be directly applied to approximate maximum likelihood estimates. An iterative Kalman smoother procedure (described in Appendix B and called IKS), in which the equation is linearly approximated[9] at each step n around previous step estimates, is therefore used to estimate the parameters $\boldsymbol{\theta}_{\mathbf{p}}$ and the tv-parameters ($\theta_{\xi,t}, \theta_{a,t}$). The procedure is initialised by pre-modelling the business cycles in a common factor Rünstler model. The procedure is iterated by applying the Kalman smoothers with diffuse initialisation at each step, until the estimates are stabilised, i.e. until the iteration error $e^{(n)}$ is lower than a fixed value. The iteration error is computed as the standard error of the difference between estimates at iteration (k) and ($k+1$).

[7] Parameters of the HP band-pass filter have been found with a formula detailed in Iacobucci (2005).

[8] There are no irregular component ($\sigma_{\varepsilon,1} = \sigma_{\varepsilon,2} = 0$), as irregular components are pre-filtered from the real data sets.

[9] For estimating non-linear models, Durbin and Koopman (2001) proposed such a method. However, they also recommend improving the estimation with importance sampling techniques. This improvement is let for further research.

3 Simulation results

The numerical properties of the estimation procedure, suggested in Section 2.5, are now analised on simulated time series. After a brief presentation of the simulation methodology (Section 3.1), the main results are detailed for a reference case in Section 3.2. Then, the influence of the parameter settings is studied in Section 3.3.

3.1 Simulation methodology

For a given vector of parameters $\boldsymbol{\theta}_{\mathbf{p}}$ and $j = 1, ..., N$, the simulation algorithm is the following:

- $(\kappa_t^{c(j)}, \kappa_t^{+c(j)}, \kappa_{2,t}^{*(j)}, \kappa_{2,t}^{+*(j)}, \delta_{2,t}^{(j)}, \gamma_{2,t}^{(j)})$ are simulated with Gaussian independent distributions.
- $(y_{1,t}^{(j)}, y_{2,t}^{(j)})$ are generated within the SCCM model with innovations computed in the previous step.
- IKS is applied to $(y_{1,t}^{(j)}, y_{2,t}^{(j)})$ and delivers estimates of parameters $(\hat{\boldsymbol{\theta}}_{\mathbf{p}}^{(j)})$ and tv-parameters $(\hat{\theta}_{\xi,t}^{(j)}, \hat{\theta}_{a,t}^{(j)})$.
- The ratio of root mean square errors (RRMSE) of the tv-parameter estimates $(\hat{x_t})$, relative to the true values (x_t), are computed with the following formula:

$$\mathsf{RRMSE}^{(j)}(x) = \frac{\sqrt{\frac{1}{n-1}\sum_{t=1}^{t=n}(x_t^{(j)} - \hat{x}_t^{(j)})^2}}{\sqrt{\frac{1}{n-1}\sum_{t=1}^{t=n}(x_t^{(j)} - E(x^{(j)}))^2}},$$

  for $x = \xi_2, a_2$. The standard error of each IKS estimate $\hat{x}_t^{(j)}$ is rescaled with the standard deviation of each tv-parameter $x_t^{(j)}$. $\mathsf{RRMSE}^{(j)}(x)$ is expected to be lower than 1. If $\mathsf{RRMSE}^{(j)}(x) > 1$, the variance of the tv-parameter x is not explained at all.

Finally, the median $\hat{\boldsymbol{\theta}}_{\mathbf{p}}$ of parameter estimates $(\hat{\boldsymbol{\theta}}_{\mathbf{p}}^{(j)})_{1\leq i\leq N}$ are computed over the N occurences (with $x = \xi_2, a_2$) together with the median RRMSE(x) based on the tv-parameters errors $(\mathsf{RRMSE}^{(j)}(x))_{1\leq i\leq N}$. The RRMSE criteria are always positive. For a given tv-parameter (x), the lower the RRMSE is, the more precise is the estimate $(\hat{x})$. A correct estimate of x is expected to have a RRMSE lower tan one, since it means that the variance of the tv-parameter x is generally not be explained at all.

3.2 Simulation results in the reference case

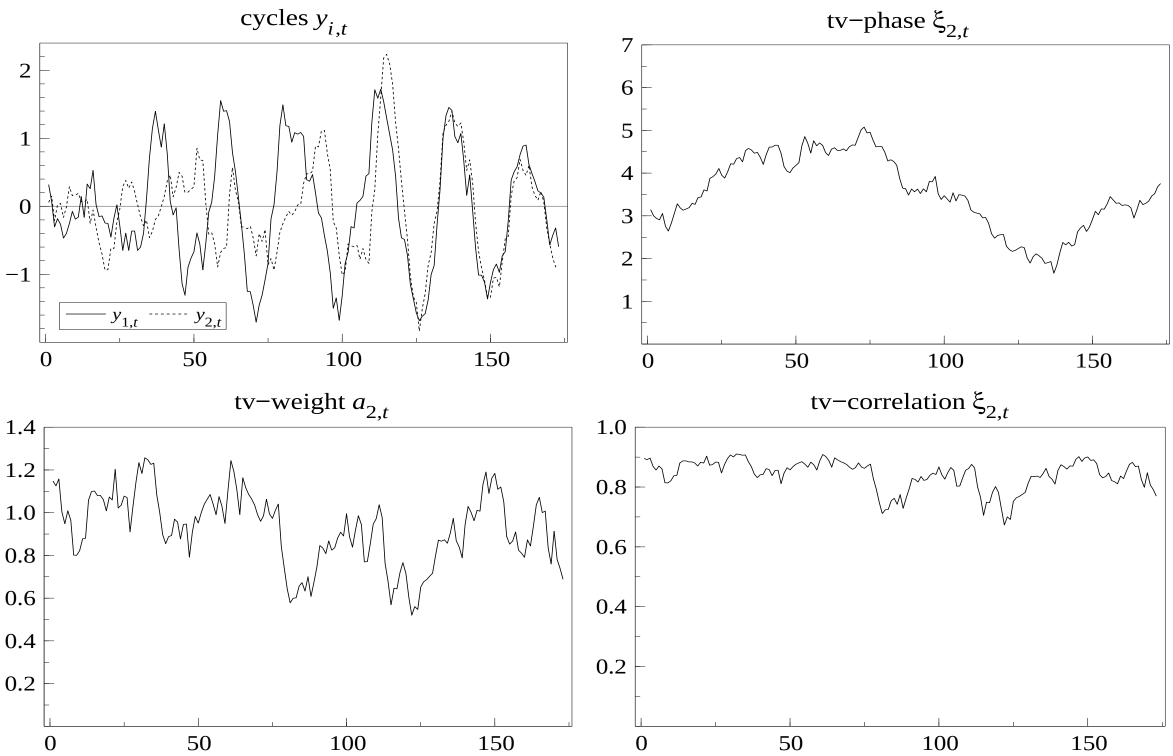
In the reference case (case 1), the vector of parameters $\mathbf{p}$ estimated in Section 4.2, for the UK and Euro-zone business cycles, is used. Common and specific cycles have the same damping factor ($\phi = 0.97$), the same frequency ($\lambda = 0.28$) and the same innovation standard deviations ($\sigma_{\kappa,c} = 0.21$ and $h_2 = 0.12$). The tv-phase and the tv-weight have the same initial value ($\xi_{2,0} =$

$3.74, a_{2,0} = 2.25$) and the same standard deviation for innovations ($\sigma_{\delta,2} = 0.15, \sigma_{\gamma,2} = 0.08$). The vector of transformed parameters is deduced from the previous values:

$$\boldsymbol{\theta}_{\mathbf{p}} = [4, 20.44, 0.12, 0.21, 0.88, 0.56, 1.25, 1.05]' .$$

In the reference case, the sample period has the same length as that in Section 4 ($n = 173$). IKS is applied on $N = 2,000$ occurences, for the parameter setting $\boldsymbol{\theta}_{\mathbf{p}}$. To give an idea how the model looks like in the reference case, an example data set is plotted. Simulated cycles and tv-parameters are reported in Figure 1.

Figure 1: Example of cycles and tv-parameters simulated with a SCCM model (case 1)



Estimation results for each simulation case are presented in Table 1. Estimates of the frequency ($\hat{\theta}_\lambda = 20.13$) and of the standard deviation of the common innovation ($\hat{\theta}_{\kappa,c} = 0.20$) are close to their true values (20.44 and 0.21), while the damping factor ($\theta_\phi = 4$) is slightly under-estimated ($\hat{\theta}_\phi = 3.51$) and the standard deviation of the specific innovation ($\theta_h = 0.12$) is slightly over-estimated ($\hat{\theta}_h = 0.15$).

The tv-parameters initial value ($\theta_{\xi,0}, \theta_{a,0}$) are well estimated, but standard deviations of tv-parameters innovations ($\theta_\delta = 0.56$ and $\theta_\gamma = 1.05$) are generally under-estimated ($\hat{\theta}_\delta = 0.09$ and $\hat{\theta}_\gamma = 0.28$). This might be related to the pile-up problem: in time-varying parameters models, when the variance of a tv-parameter (θ_δ or θ_γ) has a weak value relative to the variance of the residuals (θ_h), its maximum likelihood estimate is downward biased and has a strictly positive probability to degenerate to zero (Stock and Watson, 1998). This problem has been treated here, by using diffuse initialisation of the tv-parameters. Stock and Watson (1998) have shown that a diffuse initialisation limit the downward bias.

It may be noticed by looking at $\hat{\theta}_{\kappa,c}$, $\hat{\theta}_h$, and $\hat{\theta}_{a,0}$, that the initial value of the phase-adjusted correlation is under-estimated. Its estimate

$$\hat{\rho}_0 = \hat{a}_{2,0}/\sqrt{\hat{a}_{2,0}^2 + \hat{h}_2^2/\hat{\sigma}_{\kappa,c}^2} \tag{13}$$

is equal to 0.85, while the true value (ρ_0) is equal to 0.91. This is mainly caused by the over-estimation of the specific innovation standard deviation (h_2).

However, IKS provides satisfactory estimates of the tv-parameters, as both RRMSE are lower than one. The tv-weight estimates appear more precise than the tv-phase ones (with a RRMSE equal to 0.26 against 0.43).

The influence of the sample size on these results is analised by setting the sample size to a lower ($\underline{n} = 100$) and a larger value ($\overline{n} = 500$). A larger sample (case 1b) allows to improve estimates of the damping factor ($\hat{\theta}_\phi$), the frequency ($\hat{\theta}_\lambda$), the standard deviations of cycle innovations ($\hat{\theta}_h$ and $\hat{\theta}_{\kappa,c}$) and the initial value of the tv-weight ($\hat{\theta}_{a,0}$). Moreover, the tv-parameter estimates ($\hat{\theta}_{\xi,t}$ and $\hat{\theta}_{a,t}$) are more precise than in the reference (RRMSE are lower). Unfortunately, the pile-up problem becomes more serious: standard deviations of tv-parameters ($\hat{\theta}_\delta$ and $\hat{\theta}_\gamma$) are more under-estimated than in the reference case. For a smaller sample (case 1a), as expected, the results are the opposite.

Table 1: Parameters and RRMSE estimates of the SCCM model for various sample sizes.

Case number	Sample size (n)	Parameter estimates								RRMSE	
		$\hat{\theta}_\phi$	$\hat{\theta}_\lambda$	$\hat{\theta}_h$	$\hat{\theta}_{\kappa,c}$	$\hat{\theta}_{\xi,0}$	$\hat{\theta}_\delta$	$\hat{\theta}_{a,0}$	$\hat{\theta}_\gamma$	$\theta_{\xi,t}$	$\theta_{a,t}$
Case 1	173	3.51	20.13	0.15	0.20	0.90	0.09	1.22	0.28	0.43	0.26
Case 1a	100	3.27	20.05	0.15	0.20	0.90	0.09	1.20	0.30	0.61	0.35
Case 1b	500	3.81	20.22	0.14	0.21	0.90	0.05	1.24	0.22	0.27	0.16
Parameter values		*4.00*	*20.44*	*0.12*	*0.21*	*0.88*	*0.56*	*1.25*	*1.05*	-	-

Parameters have been estimated with the iterated Kalman smoother (IKS) procedure. The ratios of root mean square errors (RRMSE) criteria have been computed for IKS estimates of the tv-parameters.

3.3 Influence of the parameter settings on the simulation results

In order to study the simulation results sensitivity to each specific parameter, the elements of the vector $\boldsymbol{\theta}_\mathbf{p}$ have been individually changed so that sixteen further vectors of parameters have been defined. Each parameter has been set equal to values ($\underline{\theta}_p = 0.5 \times \theta_p$ and $\overline{\theta}_p = 1.5 \times \theta_p$) lower and larger than the reference one (θ_p), keeping the other parameters equal to their reference values. IKS is applied on $N = 2,000$ occurences, for each case l ($l = 1$ to 17). The estimated parameters and RRMSE are reported in Table 2.

As expected, when parameters are set to values larger than those of reference (cases 3, 5, 7, 9, 11, 13, 15 and 17), the results are the opposite of those obtained with lower values.

In cases 2, 5, 7, 8 and 14, the parameters are set close to their boundary values defined by the conditions $\phi > 0$, $0 < \lambda < \pi$ and $(\rho_t > 0, \forall t)$. In cases 2 and 5, the damping factor ϕ and the frequency λ are closer to 0 than in the reference case. In cases 7, 8 and 14, we deduce from (13) that the initial value ρ_0 of the tv-correlation is closer to 0 than in the reference case. As shown in Section 2.4, the identification conditions are necessary for identifying the tv-phase $\xi_{2,t}$. Thus, by choosing such parameters, the estimation of the transformed tv-phase $\theta_{\xi,t}$ is more difficult and its RRMSE is worse than in the reference case. For the same cases, the RRMSE of the transformed tv-weight $\theta_{a,t}$ are also generally worse than in the reference case.

In case 12, for the transformed tv-phase $\theta_{\xi,t}$, the transformed standard deviation of its innovations θ_δ is set to a lower value $\underline{\theta}_\delta$. As expected, the pile-up problem is reinforced and the standard deviation estimate $\hat{\theta}_\delta$ is more downward biased than in the reference case. The estimate $\hat{a}_{2,0}$ of the initialising value become slightly upward-biased. Other parameter estimates are not deteriorated. As the estimate $\hat{\theta}_{\xi,t}$ of the transformed tv-phase is almost constant over time, its RRMSE criterion (0.92) is worse than in the reference case, approaching the value of one. Concerning the transformed tv-weight $\theta_{a,t}$, the RRMSE is almost unchanged (0.25). The case 16 is very similar: if we set θ_γ to a lower value $\underline{\theta}_\gamma$, the pile-up problem is reinforced for the transformed tv-weight $\theta_{a,t}$ and its RRMSE is worse than in the reference.

Summing up, the tv-parameter estimates provided by IKS are all correct, because their RRMSE are all lower than one. The results are generally less precise (RRMSE are higher) than in the reference case, particularly for the tv-phase, when the parameters are set close to their boundary values (cases 2, 5, 7, 8 and 14). Because of the pile-up problem, the results are also worse, when tv-parameter innovations have low standard deviations (cases 12 and 16).

4 Empirical results

After a short presentation of the data sources, the estimated models are presented and the cyclical convergence is described with its two components, the tv-phase and the tv-correlation. In Section 4.2, we use the SCCM models outlined above to assess whether the first criterion for UK entry into the euro-zone is met, that is whether the UK business cycle has converged towards the Euro-zone cycle. In Section 4.3, using the German cycle as a reference, the UK cyclical convergence is compared with that of a major euro-zone member (France).

4.1 Data sources

The GDP time series are taken from the Eurostat database and have been retropolated to 1960 with the OECD Business Sector Database (BSDB). The series are quarterly and expressed in 1995 value of euro from 1960:1 to 2003:1. They have been seasonally adjusted and, after a logarithm transformation, they have been detrended with band-pass HP filters (parameters equal to 1 and 1,600). Because of the detrending step, results should be cautiously interpreted at the end of the

Table 2: Parameters and RRMSE estimates of the SCCM model for 17 parameter sets

Case	Parameter values								Parameter estimates								RRMSE	
number	θ_ϕ	θ_λ	θ_h	$\theta_{\kappa,c}$	$\theta_{\xi,0}$	θ_δ	$\theta_{a,0}$	θ_γ	$\hat{\theta}_\phi$	$\hat{\theta}_\lambda$	$\hat{\theta}_h$	$\hat{\theta}_{\kappa,c}$	$\hat{\theta}_{\xi,0}$	$\hat{\theta}_\delta$	$\hat{\theta}_{a,0}$	$\hat{\theta}_\gamma$	$\theta_{\xi,t}$	$\theta_{a,t}$
Case 1	4.00	20.44	0.12	0.21	0.88	0.56	1.25	1.05	3.51	20.13	0.15	0.20	0.90	0.09	1.22	0.28	0.43	0.26
Case 2	2.00	20.44	0.12	0.21	0.88	0.56	1.25	1.05	1.99	19.96	0.16	0.20	0.89	0.02	1.24	0.17	0.53	0.28
Case 3	6.00	20.44	0.12	0.21	0.88	0.56	1.25	1.05	4.55	20.15	0.15	0.20	0.90	0.13	1.21	0.35	0.42	0.26
Case 4	3.99	10.13	0.12	0.21	0.88	0.56	1.25	1.05	3.54	10.11	0.15	0.20	0.89	0.06	1.25	0.29	0.33	0.20
Case 5	3.99	30.39	0.12	0.21	0.88	0.56	1.25	1.05	3.49	30.04	0.14	0.20	0.89	0.11	1.23	0.30	0.54	0.31
Case 6	4.00	20.44	0.06	0.21	0.88	0.56	1.25	1.05	3.49	20.12	0.11	0.20	0.91	0.13	1.23	0.35	0.39	0.22
Case 7	4.00	20.44	0.18	0.21	0.88	0.56	1.25	1.05	3.54	20.16	0.20	0.20	0.89	0.05	1.21	0.23	0.49	0.32
Case 8	4.00	20.44	0.12	0.11	0.88	0.56	1.25	1.05	3.56	20.18	0.13	0.10	0.88	0.03	1.20	0.20	0.58	0.38
Case 9	4.00	20.44	0.12	0.32	0.88	0.56	1.25	1.05	3.50	20.12	0.18	0.30	0.90	0.11	1.23	0.32	0.40	0.23
Case 10	4.00	20.44	0.12	0.21	0.44	0.56	1.25	1.05	3.53	20.16	0.13	0.20	0.51	0.07	1.29	0.35	0.45	0.18
Case 11	4.00	20.44	0.12	0.21	1.32	0.56	1.25	1.05	3.51	20.12	0.16	0.20	1.29	0.10	1.06	0.23	0.44	0.35
Case 12	4.00	20.44	0.12	0.21	0.88	0.28	1.25	1.05	3.52	20.14	0.15	0.20	0.91	0.01	1.29	0.29	0.92	0.25
Case 13	4.00	20.44	0.12	0.21	0.88	0.84	1.25	1.05	3.49	20.15	0.15	0.20	0.88	0.18	1.12	0.29	0.30	0.28
Case 14	4.00	20.44	0.12	0.21	0.88	0.56	0.63	1.05	3.52	20.16	0.13	0.20	0.88	0.08	0.58	0.25	0.65	0.22
Case 15	4.00	20.44	0.12	0.21	0.88	0.56	1.88	1.05	3.52	20.13	0.18	0.20	0.90	0.08	1.86	0.27	0.37	0.39
Case 16	4.00	20.44	0.12	0.21	0.88	0.56	1.25	0.53	3.55	20.16	0.15	0.20	0.90	0.06	1.23	0.10	0.39	0.66
Case 17	4.00	20.44	0.12	0.21	0.88	0.56	1.25	1.58	3.48	20.13	0.15	0.20	0.89	0.13	1.19	0.47	0.49	0.17

Parameters have been estimated with the iterated Kalman smoother (IKS) procedure. The ratios of root mean square error (RRMSE) criteria have been computed for IKS estimates of the tv-parameters.

sample[10], i.e. from 2000:1 to 2003:1. Estimation of SCCM models has been carried out using algorithms and routines written on EViews.

4.2 On the convergence of UK and euro-zone cycles

The bivariate SCCM model has been estimated with the IKS procedure applied to the UK and Euro-zone detrended GDP series (UK-EU case). The euro-zone cycle is chosen as the reference cycle. Parameter estimates are reported in Table 3 and tv-parameter estimates are presented in the Figure 2. Cycle parameters estimates (ϕ and λ) are standard for HP filtered GDP series. In particular, the estimated frequency implies a period equal to 22 quarters, i.e. 6 years.

As regards the tv-phase estimates, they are always positive, with an average equal to 3 quarters: these values are lower during the 1970s, higher at the beginning of the 1980s and they decrease towards the end of the sample. UK recessions tend to be synchronised with that of the euro-zone during the 1970s, because of major symmetric oil shocks. At the beginning of the 1980s, UK recovery leads that of the euro-zone by one year. Thereafter, the euro-zone slowdown occurred almost 1.5 years later than that of the UK (in 1993) because of the German re-unification. Finally, after the high growth at the end of the 1990s, the United Kingdom and the euro-zone both experienced a slowdown in 2001. The UK-German synchronisation seems to have been reinforced towards the end of the 1990s.

Concerning the tv-correlation estimates, after some strong movements in the 1960s, an increase occurs during the 1970s (symmetric oil shock) and the tv-correlation remains quite stable and very high (around 90%) during the 1980s. However, in the 1990s, despite the cyclical synchronisation, the tv-correlation decreases (towards 60% in 2003:1), with the UK cycle particularly dampened relative to that of the euro-zone. This may be related to fiscal policies differences: the UK public deficits has been contra-cyclical, whereas that of the euro-zone has not (Barrell and Weale, 2003). During this period, contrary to the "golden rule" followed by the UK government, the euro-zone governments have followed a deficit rule (defined by the Growth and Stability Pact) that did not take into account the timing of business cycles.

Until the end of the 1990s, the results are generally consistent with the previous papers, which used a similar approach of the cyclical convergence (Belo, 2001 and Koopman and Azevedo, 2004), but the approach is improved at the end of the 1990s. With simple correlation coefficients, Belo (2001) has also shown a persisting lead of the UK cycle relative to that of the euro-zone and, after a correction for this lead, a strong association between the two cycles. These results have been confirmed by Koopman and Azevedo (2004) with an extended unobserved component model. However, as explained previously, Koopman and Azevedo (2004) model monotonous evolution of the phase and the phase-adjusted correlation. Their model does not allow for a succession of convergence and divergence movements. Thus, in the UK-EU case, they estimate a global increase of the phase-adjusted correlation since 1970 and do not detect the recent decrease we show in this

[10]Such a filter, as other univariate methods, provide very uncertain estimates at the end of the sample (Orphanides and van Norden, 2002). For the euro area, Rünstler (2002) has shown that cycle estimates provided by univariate methods, in particular the HP filter, are uncertain during the last three years of the sample.

section. This decrease was also undetected by Belo (2001). He computed moving-average indicators with a window of 12 years. With the annual sample 1960-1999, such indicators become useless for detecting any variation occuring after 1993 ($12/2 = 6$ years before the end of the sample).

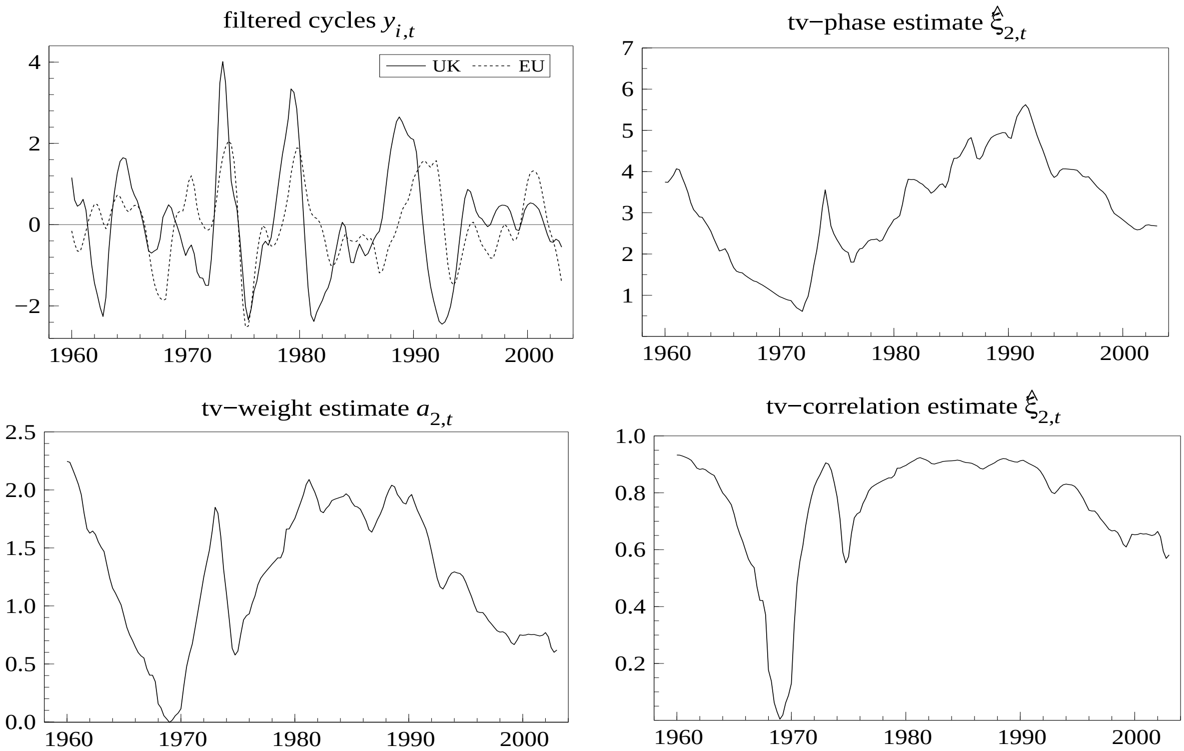
Returning to the issue raised by the UK government about entering the euro-zone, the estimated SCCM model shows mixed results: an improved synchronisation between UK and EU cycles but also a decreased correlation. UK entry would thus require an improvement in fiscal coordination, in order to deal with persistent asymmetric shocks.

Table 3: Parameters estimates for UK–EU

Data set	$\hat{\phi}$	$\hat{\lambda}$	$\hat{h}_2$	$\hat{\sigma}_{\kappa,c}$	$\hat{\xi}_{2,0}$	$\hat{\sigma}_{\delta,2}$	$\hat{a}_{2,0}$	$\hat{\sigma}_{\gamma,2}$
UK–EU	0.97	0.28	0.12	0.21	3.74	0.15	2.25	0.08

Parameter estimates ($\hat{\mathbf{p}}$) have been deduced from those ($\hat{\boldsymbol{\theta}}_{\mathbf{p}}$) estimated with the iterated Kalman smoother (IKS) procedure.

Figure 2: Empirical results for UK-EU



4.3 On the convergence of French, UK and German cycles

One may go on asking whether the UK convergence process differs from that of the euro-zone countries. A brief comparison will be conducted with the French case. A direct bias may affect such a comparison: UK does not belong to the Euro-zone aggregate, contrary to France. This difference might induce an artificial over-estimation of synchronisation and correlation, in the French case regard to that of the UK. Therefore, the euro-zone cycle is not chosen as reference cycle and is replaced by that of Germany, which is the biggest country of the euro-zone and has been the currency anchor during the European Currency Mechanism (ECM) period (1979-1999). Two SCCM models are estimated: one for the couple United Kingdom/Gemany (UK-GE) and one for the couple France/Germany (FR-GE). The parameter estimates are reported in Table 4. Cycles and tv-parameter estimates are presented in Figure 3

For the UK-GE case, the results are very similar to those described in the UK-EU case (Section 4.2). Indeed, the German cycle is the major component of the Euro-zone cycle (Bentoglio, Fayolle, and Lemoine, 2001). However, the UK-GE tv-correlation decrease occurs later and is less strong than in the UK-EU case. The tv-correlation decrease begins in the mid-1990s with a fall to 0.8 in 2003:1.

In the FR-GE case, tv-parameters are very different. The tv-phase is always lower for FR-GE than for UK-GE. In the 1970s, the French cycle generally had a lag of one or two quarters relative to that of Germany. A small lead (around two quarters) appeared at the beginning of the 1990s, because of the reunification, and this lead has dwindled away since the mid-1990s.

The phase-adjusted tv-correlation is high in the 1970s (oil shocks). Isolated economic policies have led to a fall in the tv-correlation at the beginning of the 1980s. Since then, a political reorientation towards Europe has implied a convergence, which appears in the form of a tv-correlation increase towards 0.95 in 2003:1. Unlike the UK, France and Germany experienced strong fluctuations between 1997 and 2003 and their governments amplified these fluctuations with pro-cyclical fiscal policies.

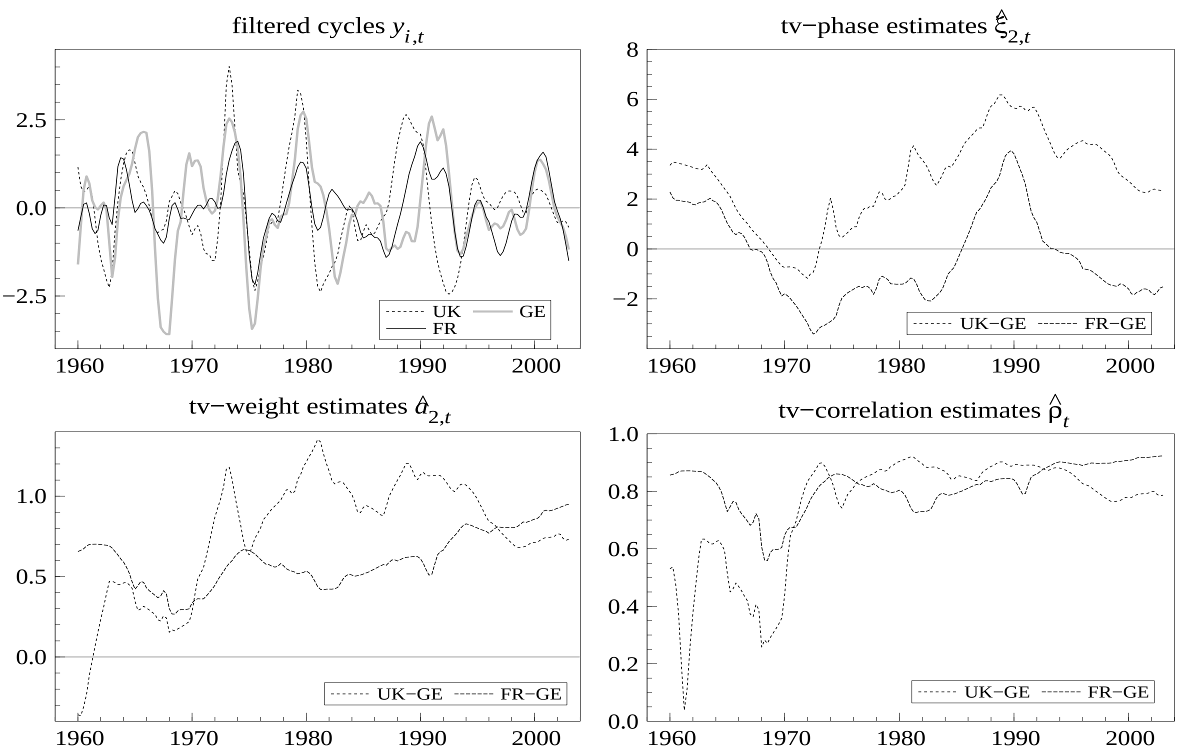
In terms of synchronisation and correlation, the degree of convergence between French and German cycles appears to be very strong since 1995, whereas a correlation divergence has emerged between UK and Germany.

Table 4: Parameters estimates for FR–GE and UK–GE

Data set	$\hat{\phi}$	$\hat{\lambda}$	$\hat{h}_2$	$\hat{\sigma}_{\kappa,c}$	$\hat{\xi}_{2,0}$	$\hat{\sigma}_{\delta,2}$	$\hat{a}_{2,0}$	$\hat{\sigma}_{\gamma,2}$
UK–GE	0.97	0.29	0.11	0.31	1.72	0.13	0.66	0.02
FR–GE	0.97	0.29	0.15	0.31	3.38	0.20	0.37	0.04

Parameter estimates ($\hat{\mathbf{p}}$) have been deduced from those ($\hat{\boldsymbol{\theta}}_{\mathbf{p}}$) estimated with the iterated Kalman smoother (IKS) procedure.

Figure 3: Empirical results for FR–GE and UK–GE



5 Conclusion

The SCCM model is an extension of the models developed by Rünstler (2004) and Koopman and Azevedo (2004). The main contribution of the paper concerns the dynamic property of the SCCM model: it formalises the stochastic evolution of the relation between two cycles, which is characterised by the tv-phase and the phase-adjusted tv-correlation. An iterated Kalman Smoother (IKS) procedure is proposed and checked on various simulated data sets, showing that the IKS procedure provides estimates that capture correctly the dynamics of the tv-parameters. However, the variance of the tv-parameters is downward biased, because of the pile-up problem that was detailed for the general TVP models in Stock and Watson (1998). An analysis of the influence of the parameter settings on the simulation results shows that the tv-phase estimates are less precise, when the parameters are set close to their boundary values. Such a problem should also occur when estimating the phase-shift with the models of Rünstler (2004) and Koopman and Azevedo (2004). The estimation procedure assessment was not carried out in these papers.

The paper also provides some empirical results. SCCM models have been applied on British, French, German and euro-zone cycles, from 1960:1 to 2003:1. During the 1970s, oil shocks imply synchronised recessions with high correlations. From 1980 to 1995, the United Kingdom has known a substantial lead relative to the euro-zone, despite highly correlated shocks. Since the mid-1990s, the UK cycle has been considerably dampened and become progressively more synchronised with the euro-zone cycle. Hence, contrary to France, the two criteria are not validated: the synchronisation convergence has improved, but was not fully achieved; after a correlation convergence in the 1970s, a correlation divergence occured during the 1990s. The recent correlation divergence is impossible to detect with the methodologies employed in some recent papers (Koopman and Azevedo, 2004 and Wynne and Koo, 2000). This divergence can have important consequences: as the risk of asymmetric shocks remains, the entry of the United Kingdom into the Euro-zone would require improved fiscal coordination.

Although our convergence model has proved to be an interesting starting point for testing the stochastic convergence of business cycles in a probabilistic framework, some improvements have been left for future research. The estimation procedure of our non-linear model could be made more precise by using importance sampling techniques. In particular, such techniques would make it possible to simulate posterior distribution of the tv-parameters and to estimate their confidence bands. The model could be applied with a higher multivariate dimension. An explanation of the convergence could be given by integrating exogenous variables, like economic policiy indicators (interest rates, exchange rates and fiscal indicators) in the convergence mechanism.

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Appendices

A Auto-covariance of business cycles in a SCCM model

A.1 Simple multivariate form of the SCCM model

The SCCM model (8) is statistically equivalent (has the same cross-autocovariance functions) to the following model:

$$
\begin{aligned}
y_{1,t} &= [1,0]\,\bar{\boldsymbol{\psi}}_{1,t} + \varepsilon_{1,t},\\
y_{2,t} &= \left[\cos(\lambda\xi_{2,t}), \sin(\lambda\xi_{2,t})\right]\bar{\boldsymbol{\psi}}_{2,t} + \varepsilon_{2,t},\\
\bar{\boldsymbol{\psi}}_{i,t} &= \phi T_\lambda \bar{\boldsymbol{\psi}}_{i,t-1} + \bar{\boldsymbol{\kappa}}_{i,t},
\end{aligned}
$$

with $\bar{\boldsymbol{\psi}}_{1,t} = \left[\psi_{i,t}, \psi^+_{i,t}\right]'$ and $\bar{\boldsymbol{\kappa}}_{i,t} = \left[\kappa_{i,t}, \kappa^+_{i,t}\right]'$. The initial conditions at $t = 0$ are defined by $\bar{\boldsymbol{\psi}}_{i,0} = \bar{\boldsymbol{\kappa}}_{i,0}$. $\boldsymbol{\kappa}_t = \left[\kappa_{1,t}, \kappa_{2,t}\right]'$ and $\boldsymbol{\kappa}^+_t = \left[\kappa^+_{1,t}, \kappa^+_{2,t}\right]'$ are bivariate normal disturbances mutually uncorrelated at all time periods and have covariance matrices $\Sigma_{\kappa,t}$:

$$
\begin{bmatrix} \kappa_{1,t} \\ \kappa_{2,t} \end{bmatrix} \sim NID(0,\Sigma_\kappa), \begin{bmatrix} \kappa^+_{1,t} \\ \kappa^+_{2,t} \end{bmatrix} \sim NID(0,\Sigma_\kappa), \text{ with } \Sigma_{\kappa,t} = \begin{bmatrix} \sigma^2_{\kappa,1} & \rho_t\sigma_{\kappa,1}\sigma_{\kappa,2} \\ \rho_t\sigma_{\kappa,1}\sigma_{\kappa,2} & \sigma^2_{\kappa,2} \end{bmatrix}.
$$

$\bar{\boldsymbol{\kappa}}_{i,t}$ are bivariate normal disturbances and have 2×2 covariance matrices $\sigma^2_{\kappa,i} I_2$ (with I_2 the 2×2 identity matrix).

Given that

$$
E\left(\psi_{1,0}\psi_{2,0}\middle|\,\boldsymbol{\rho}_{1:t}\right) = E\left(\kappa_{1,0}\kappa_{2,0}\middle|\,\boldsymbol{\rho}_{1:t}\right) = E\left(\kappa^+_{1,0}\kappa^+_{2,0}\middle|\,\boldsymbol{\rho}_{1:t}\right) = E\left(\psi^+_{1,0}\psi^+_{2,0}\middle|\,\boldsymbol{\rho}_{1:t}\right),
$$

we can prove recursively that

$$
E\left(\psi_{1,t}\psi_{2,t}\middle|\,\boldsymbol{\rho}_{1:t}\right) = E\left(\psi^+_{1,t}\psi^+_{2,t}\middle|\,\boldsymbol{\rho}_{1:t}\right) \tag{A.1}
$$

for each t.

A.2 Cross-autocovariance between bivariate cycles

Before considering the covariance matrix of business cycles, conditional on the tv-parameters, sinc $(T_\lambda)^\tau = T_{\lambda\tau}$ for all integer τ, we compute the cross-autocovariance $\Omega_\rho(\tau,t)$ between the bivariate cycles $\bar{\boldsymbol{\psi}}_{1,t}$ and $\bar{\boldsymbol{\psi}}_{2,t-\tau}$ at a lag $\tau > 0$, conditional on $\boldsymbol{\rho}_{1:t} = [\rho_1, ..., \rho_t]'$:

$$
\begin{aligned}
\Omega_\rho(\tau,t) &= E\left(\bar{\boldsymbol{\psi}}_{1,t}\bar{\boldsymbol{\psi}}'_{2,t-\tau}\middle|\,\boldsymbol{\rho}_{1:t}\right) \\
&= E\left[\left(\phi^\tau T_{\lambda\tau}\bar{\boldsymbol{\psi}}_{1,t-\tau} + \sum_{i=0}^{\tau-1}\phi^i T_{\lambda i}\bar{\boldsymbol{\kappa}}_{1,t-i}\right)\bar{\boldsymbol{\psi}}'_{2,t-\tau}\middle|\,\boldsymbol{\rho}_{1:t}\right] \\
&= \phi^\tau T_{\lambda\tau}\Omega_\rho(0,t-\tau),
\end{aligned} \tag{A.2}
$$

with $E\left[\cdot\middle|\,\boldsymbol{\rho}_{1:t}\right]$ the “conditional on $\boldsymbol{\rho}_{1:t}$” expectation. Given that $\bar{\boldsymbol{\psi}}_{1,t-1}\perp\bar{\boldsymbol{\kappa}}_{2,t}$ and $\bar{\boldsymbol{\psi}}_{2,t-1}\perp\bar{\boldsymbol{\kappa}}_{1,t}$, the cross-covariance $\Omega_\rho(0,t)$ between $\bar{\boldsymbol{\psi}}_{1,t}$ and $\bar{\boldsymbol{\psi}}_{2,t}$, conditional on $\boldsymbol{\rho}_{1:t}$, is given by

$$
\begin{aligned}
\Omega_\rho(0,t) &= E\left(\bar{\boldsymbol{\psi}}_{1,t}\bar{\boldsymbol{\psi}}'_{2,t}\middle|\,\boldsymbol{\rho}_{1:t}\right) \\
&= E\left[\left(\phi T_\lambda\bar{\boldsymbol{\psi}}_{1,t-1} + \bar{\boldsymbol{\kappa}}_{1,t}\right)\left(\phi T_\lambda\bar{\boldsymbol{\psi}}_{2,t-1} + \bar{\boldsymbol{\kappa}}_{2,t}\right)'\middle|\,\boldsymbol{\rho}_{1:t}\right] \\
&= \phi^2 T_\lambda\Omega_{\boldsymbol{\rho}}(0,t-1)T'_\lambda + \rho_t\sigma_{\kappa,1}\sigma_{\kappa,2}I_2,
\end{aligned}
$$

Since $\psi_{1,t}^{+} \perp \psi_{2,t}$, $\psi_{1,t} \perp \psi_{2,t}^{+}$ and from (A.1), it follows that

$$\Omega_\rho(0,t) = \begin{bmatrix} E\left(\psi_{1,t}\psi_{2,t}\middle|\,\boldsymbol{\rho}_{1:t}\right) & E\left(\psi_{1,t}\psi_{2,t}^{+}\middle|\,\boldsymbol{\rho}_{1:t}\right) \\ E\left(\psi_{1,t}^{+}\psi_{2,t}\middle|\,\boldsymbol{\rho}_{1:t}\right) & E\left(\psi_{1,t}^{+}\psi_{2,t}^{+}\middle|\,\boldsymbol{\rho}_{1:t}\right) \end{bmatrix} = E\left(\psi_{1,t}\psi_{2,t}\middle|\,\boldsymbol{\rho}_{1:t}\right) I_2$$

Hence, $\Omega_\rho(0,t-1)$ commutes with the matrix T_λ which is orthogonal ($T_\lambda T_\lambda^{'} = I_2$),

$$\Omega_\rho(0,t) = \phi^2\Omega_\rho(0,t-1) + \rho_t\sigma_{\kappa,1}\sigma_{\kappa,2}I_2. \tag{A.3}$$

Since $\Omega_\rho(0,0) = \rho_0\sigma_{\kappa,1}\sigma_{\kappa,2}I_2$ and given equation (A.3),

$$\Omega_\rho(0,t) = (1-\phi^2)^{-1}\tilde{\rho}_t\sigma_{\kappa,1}\sigma_{\kappa,2}I_2, \tag{A.4}$$

with $\tilde{\rho}_t = (1-\phi^2)\sum_{i=0}^{t}\phi^{2i}\rho_{t-i}$ the exponential smoother of ρ_t (the parameter is equal to ϕ^2).

From equations (A.2) and (A.4), we derive the cross-autocovariance $\Omega_\rho(\tau,t)$ between bivariate cycles $\bar{\boldsymbol{\psi}}_{1,t}$ and $\bar{\boldsymbol{\psi}}_{2,t-\tau}$ at a lag $\tau > 0$, conditional on $\boldsymbol{\rho}_{1:t} = [\rho_1, ..., \rho_t]'$:

$$\Omega_\rho(\tau,t) = (1-\phi^2)^{-1}\phi^\tau\tilde{\rho}_{t-\tau}\sigma_{\kappa,1}\sigma_{\kappa,2}T_{\lambda\tau}, \tag{A.5}$$

A.3 Auto-covariance matrix between business cycles

From (A.5), the cross-autocovariance $\gamma_{\rho,\xi}(\tau,t)$ between business cycles $y_{1,t}$ and $y_{2,t}$ at a lag $\tau > 0$, conditional on $\boldsymbol{\rho}_{1:t}$ and $\boldsymbol{\xi}_{1:n} = [\xi_1, ..., \xi_n]'$, is given by

$$\begin{aligned}
\gamma_{\rho,\xi}(\tau,t) &= E\left[\left[\begin{array}{cc} 1 & 0 \end{array}\right]\bar{\boldsymbol{\psi}}_{1,t}\bar{\boldsymbol{\psi}}_{2,t-\tau}^{'}\begin{bmatrix}\cos\xi_{2,t-\tau}\\ \sin\xi_{2,t-\tau}\end{bmatrix}\middle|\,\boldsymbol{\rho}_{1:t},\boldsymbol{\xi}_{1:n}\right] \\
&= \left[\begin{array}{cc} 1 & 0 \end{array}\right]\Omega_\rho(\tau,t)\begin{bmatrix}\cos\xi_{2,t-\tau}\\ \sin\xi_{2,t-\tau}\end{bmatrix} \\
&= (1-\phi^2)^{-1}\phi^\tau\tilde{\rho}_{t-\tau}\sigma_{\kappa,1}\sigma_{\kappa,2}\cos\left[\lambda(\tau-\xi_{2,t-\tau})\right].
\end{aligned}$$

with $E(\cdot|\,\boldsymbol{\rho}_{1:t},\boldsymbol{\xi}_{1:t})$ the "conditional on $\boldsymbol{\rho}_{1:t}$ and $\boldsymbol{\xi}_{1:n}$" expectation.

By using similar arguments, it might be proved more generally that the autocovariance matrix $\Gamma_{\rho,\xi}(\tau,t)$ of $[\psi_{1,t};\cos(\lambda\xi_{2,t})\psi_{2,t} + \sin(\lambda\xi_{2,t-\tau})\psi_{2,t}^{+}]'$ at a lag $\tau \geq 0$, conditional on $\boldsymbol{\rho}_{1:t}$ and $\boldsymbol{\xi}_{1:n}$, is given by

$$\Gamma_{\rho,\xi}(\tau,t) = (1-\phi^2)^{-1}\phi^\tau\begin{bmatrix}\sigma_{\kappa,1}^2\cos(\lambda\tau) & \tilde{\rho}_{t-\tau}\sigma_{\kappa,1}\sigma_{\kappa,2}\cos\left[\lambda(\tau+\xi_{2,t})\right] \\ \tilde{\rho}_{t-\tau}\sigma_{\kappa,1}\sigma_{\kappa,2}\cos\left[\lambda(\tau-\xi_{2,t-\tau})\right] & \sigma_{\kappa,2}^2\cos(\lambda\tau)\end{bmatrix};$$

at a lag $\tau < 0$, it can also be shown that

$$\Gamma_{\rho,\xi}(\tau,t) = (1-\phi^2)^{-1}\phi^{-\tau}\begin{bmatrix}\sigma_{\kappa,1}^2\cos(\lambda\tau) & \tilde{\rho}_{t}\sigma_{\kappa,1}\sigma_{\kappa,2}\cos\left[\lambda(\tau+\xi_{2,t})\right] \\ \tilde{\rho}_{t}\sigma_{\kappa,1}\sigma_{\kappa,2}\cos\left[\lambda(\tau-\xi_{2,t-\tau})\right] & \sigma_{\kappa,2}^2\cos(\lambda\tau)\end{bmatrix}.$$

At time t, if $\phi = 0$, $\phi = 1$, $\lambda = 0$, $\lambda = \pi$ or $\boldsymbol{\rho}_{1:t} = \mathbf{0}_t$, the autocovariance matrix $\Gamma_{\rho,\xi}(\tau,t)$ does not depend on $\xi_{2,t}$ and the phase-shift $\xi_{2,t}$ is not identifiable. Thus, the *identification conditions* ($\phi > 0$), ($0 < \lambda < \pi$) and ($\rho_t > 0$, $\forall t$) are assumed.

B Iterative Kalman smoother (IKS)

The iterative Kalman smoother (IKS) is an iterative procedure, designed for estimating a linear approximate of the SCCM model. The notations are defined in Section 2.4 and Section 2.5.

Initialisation $k = 0$

- Estimation of $\bar{\boldsymbol{\psi}}_t^c, \bar{\boldsymbol{\psi}}_{2,t}^*, \hat{\theta}_\xi, \hat{\theta}_a$ in the common factor Rünstler model, by applying the Kalman and EM algorithms to the equation system (6), where transformed parameters $\boldsymbol{\theta}_{\mathbf{p}}$ have been integrated. The phase-shift θ_ξ and the weight a are also deduced from transformed parameters: $(\xi = \theta_\xi, a = \theta_a)$
- $\hat{\theta}_{\xi,t}^{(0)} = \hat{\theta}_\xi, \hat{\theta}_{a,t}^{(0)} = \hat{\theta}_a$.
- $e^{(0)} = 1$.

Iteration k+1

- If the iteration error $e^{(k)} > 10^{-7}$, Kalman and EM algorithms (with diffuse initialisation) are applied to models (B.1) and (B.2), otherwise the algorithm is stopped. First, $a_t^{(k+1)}$ is estimated conditional to $\hat{\xi}_t^{(k)}$:

$$\left\{\begin{array}{l} y_{1,t} = [1,0]\bar{\boldsymbol{\psi}}_t^c \\ y_{2,t} = \theta_{a,t}^{(k+1)}\left[\cos\hat{\theta}_{\xi,t}^{(k)}, \sin\hat{\theta}_{\xi,t}^{(k)}\right]\bar{\boldsymbol{\psi}}_t^c + [1,0]\tilde{\psi}_{2,t}^{*(k+1)} \\ \bar{\boldsymbol{\psi}}_{2,t}^{*(k+1)} = \frac{|\theta_\phi|}{\sqrt{1+\theta_\phi^2}} T_{\frac{2\pi}{2+|\theta_\lambda|}} \bar{\boldsymbol{\psi}}_{2,t-1}^{*(k+1)} + \tilde{\kappa}_{2,t}^* \\ \theta_{a,t}^{(k+1)} = \theta_{a,t-1}^{(k+1)} + \gamma_{2,t}, \theta_{a,0}^{(k+1)} = \theta_{a,0} \end{array}\right. . \tag{B.1}$$

with $\kappa_t^c \sim N(0,\theta_{\kappa,c})$, $\kappa_t^{+c} \sim N(0,\theta_{\kappa,c})$, $\kappa_{2,t}^* \sim N(0,\theta_h)$, $\kappa_{2,t}^{+*} \sim N(0,\theta_h)$, $\gamma_{2,t} \sim N(0, \frac{\theta_\gamma}{\sqrt{n}})$.

- Then, $\xi_t^{(k+1)}$ is estimated conditional to $\hat{a}_t^{(k+1)}$ and $\hat{\xi}_t^{(k)}$:

$$\left\{\begin{array}{l} y_{1,t} = [1,0]\bar{\boldsymbol{\psi}}_t^c \\ \tilde{y}_{2,t} = \hat{\theta}_{a,t}^{(k+1)}\hat{\theta}_{\xi,t}^{(k+1)}\left[-\sin\hat{\theta}_{\xi,t}^{(k)}, \cos\hat{\theta}_{\xi,t}^{(k)}\right]\bar{\boldsymbol{\psi}}_{1,t} + [1,0]\tilde{\psi}_{2,t}^{**(k+1)} \\ \bar{\boldsymbol{\psi}}_{2,t}^{**(k+1)} = \frac{|\theta_\phi|}{\sqrt{1+\theta_\phi^2}} T_{\frac{2\pi}{2+|\theta_\lambda|}} \bar{\boldsymbol{\psi}}_{2,t-1}^{**(k+1)} + \tilde{\kappa}_{2,t}^* \\ \theta_{\xi,t}^{(k+1)} = \theta_{\xi,t-1}^{(k+1)} + \frac{2\pi}{2+|\theta_\lambda|}\delta_{2,t}, \theta_{\xi,0}^{(k+1)} = \theta_{\xi,0} \end{array}\right. , \tag{B.2}$$

with $\kappa_t^c \sim N(0,\theta_{\kappa,c})$, $\kappa_t^{+c} \sim N(0,\theta_{\kappa,c})$, $\kappa_{2,t}^* \sim N(0,\theta_h)$, $\kappa_{2,t}^{+*} \sim N(0,\theta_h)$, $\delta_{2,t} \sim N(0, \frac{2+|\theta_\lambda|}{2\pi}\frac{\theta_\delta}{\sqrt{n}})$, $\tilde{\psi}_{2,t}^{**(k+1)}$ the specific cycle of the linearised model (B.2) at the iteration $(k+1)$ and $\tilde{y}_{2,y}$ the adjusted series defined by:

$$\tilde{y}_{2,t} = y_{2,t} - \hat{\theta}_{a,t}^{(k+1)}\left[\cos\hat{\theta}_{\xi,t}^{(k)}, \sin\hat{\theta}_{\xi,t}^{(k)}\right]\bar{\boldsymbol{\psi}}_{1,t} + \hat{\theta}_{a,t}^{(k+1)}\hat{\theta}_{\xi,t}^{(k)}\left[-\sin\hat{\theta}_{\xi,t}^{(k)}, \cos\hat{\theta}_{\xi,t}^{(k)}\right]\bar{\boldsymbol{\psi}}_{1,t}.$$

- Finally, the iteration error $e^{(k+1)}$ is computed:

$$e^{(k+1)} = 0.5 \times \sqrt{\frac{1}{n}\sum_{t=0}^{n}\left(\hat{\theta}_{a,t}^{(k+1)} - \hat{\theta}_{a,t}^{(k)}\right)^2} + 0.5 \times \sqrt{\frac{1}{n}\sum_{t=0}^{n}\left(\hat{\theta}_{\xi,t}^{(k+1)} - \hat{\theta}_{\xi,t}^{(k)}\right)^2}.$$